

# ON FINITE MARKED LENGTH SPECTRAL RIGIDITY OF HYPERBOLIC CONE SURFACES AND THE THURSTON METRIC

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**ABSTRACT.** We study the geometry of hyperbolic cone surfaces, possibly with cusps or geodesic boundaries. We prove that any hyperbolic cone structure on a surface of non-exceptional type is determined up to isotopy by the geodesic lengths of a finite specific homotopy classes of non-peripheral simple closed curves. As an application, we show that the Thurston asymmetric metric is well-defined on the Teichmüller space of hyperbolic cone surfaces with fixed cone angles and boundary lengths. We compare such a Teichmüller space with the Teichmüller space of complete hyperbolic surfaces with punctures, by showing that the two spaces (endowed with the Thurston metric) are almost isometric.

**Keywords:** finite marked length spectral rigidity; Teichmüller space; hyperbolic cone surfaces; Thurston metric.

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## 1. INTRODUCTION

Let  $S = S_{g,n}$  be an oriented surface of genus  $g$  with  $n$  boundary components. We shall consider hyperbolic cone metrics on  $S$ , that is, hyperbolic structure on  $S$  such that each boundary component is either a cone point (with cone angle strictly less than  $\pi$ ), a puncture (i.e., a cone point with zero angle), or a simple closed geodesic with positive length. In this case, a boundary component  $\Delta$  of  $S$  is associated with a *generalized length function* (also called *boundary assignment*)  $\lambda(\Delta)$ , defined by

$$(1) \quad \lambda(\Delta) = \begin{cases} -\theta, & \text{if } \Delta \text{ is a cone point of angle } \theta \in (0, \pi), \\ 0, & \text{if } \Delta \text{ is a cusp,} \\ l, & \text{if } \Delta \text{ is a geodesic boundary of length } l > 0. \end{cases}$$

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**1.1. Parametrization of Teichmüller space.** Let  $\mathcal{T}_{g,n}$  be the Teichmüller space of marked hyperbolic cone metrics on  $S$ . Given any  $\Lambda = (\lambda_1, \dots, \lambda_n) \in (-\pi, \infty)^n$ , we let  $\mathcal{T}_{g,n}(\Lambda)$  be the subspace of  $\mathcal{T}_{g,n}$  corresponding to hyperbolic cone surfaces whose boundary components have fixed generalized lengths  $\Lambda$ .

It is well known [7] that the Teichmüller space  $\mathcal{T}_{g,n}(0)$  can be parameterized by the geodesic lengths of finitely many specified simple closed curves. Moreover, Seppälä and Sorvali [24] proved that only  $6g - 6 + 2n$  simple closed curves are required if  $n \neq 0$ . (For a closed surface of genus  $g$ ,  $6g - 5$  simple closed curves are needed, see Schmutz [22].) A natural question is whether these results still hold for our enlarged Teichmüller space  $\mathcal{T}_{g,n}$ . In §4.1, we prove that:

**Theorem A .** Let  $S_{g,n}$  be a surface with  $g \geq 1, n \geq 1$  or  $g = 0, n \geq 6$ . The Teichmüller space  $\mathcal{T}_{g,n}$  can be parameterized by the geodesic lengths of finitely many non-peripheral simple closed curves (see Definition 2) of  $S_{g,n}$ . Moreover, the minimal number of these parameters is less than  $12g - 12 + 32n$ .

If  $(g, n) \in \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (1, 0)\}$ , we call the surface  $S_{g,n}$  *exceptional*. Otherwise, we call surface  $S_{g,n}$  *non-exceptional*.

**1.2. Comparisons of geometry between hyperbolic cone surfaces.** Intuitively a cusp can be considered as a limit of cone points with angles tending to zero, or as a limit of geodesic boundaries with lengths tending to zero. We want to compare the geometries of hyperbolic cone surfaces under different assignments of cone angles and boundary lengths. In particular, we want to compare the geodesic lengths of simple closed curves on hyperbolic cone surfaces when we modify cone points or geodesic boundaries to cusps.

For a given pants decomposition  $\Gamma = \{\gamma_1, \dots, \gamma_{3g-3+n}\}$  and a collection of seams  $\mathcal{B} = \{\beta_1, \dots, \beta_k\}$ , we define  $F_{\Gamma, \mathcal{B}}$  by

$$\begin{aligned} F_{(\Gamma, \mathcal{B})} : \mathcal{T}_{g,n} &\rightarrow \mathcal{T}_{g,n}(0) \\ (\Lambda, L, T) &\mapsto (0, L, T), \end{aligned}$$

where  $(\Lambda, L, T)$  and  $(0, L, T)$  are the corresponding Fenchel-Nielsen (length-twist) coordinates (see Section 3). We denote the restriction of  $F_{\Gamma, \mathcal{B}}$  on  $\mathcal{T}_{g,n}(\Lambda)$  by  $F_{\Gamma, \mathcal{B}, \Lambda}$ .

**Theorem B** (Length comparison inequalities). Let  $S$  be a non-exceptional surface,  $\Gamma = \{\gamma_1, \dots, \gamma_{3g-3+n}\}$  be a pants decomposition and  $\mathcal{B} = \{\beta_1, \dots, \beta_k\}$  be a collection of seams. There exist constants  $C, D$  depending on  $\Lambda$  such that for any  $X \in \mathcal{T}_{g,n}(\Lambda)$ ,  $X' = F_{\Gamma, \mathcal{B}, \Lambda}(X)$  and any

isotopy class of non-peripheral simple closed curve  $[\alpha]$ ,

$$\begin{cases} |l_X([\alpha]) - l_{X'}([\alpha])| \leq D \sum_{j=1}^{3g-3+n} i([\alpha], [\gamma_j]), \\ \frac{1}{C} \leq \frac{l_X([\alpha])}{l_{X'}([\alpha])} \leq C, \end{cases}$$

where  $i(\cdot, \cdot)$  is the geometric intersection number, and  $l_X([\alpha])$  is the length of the geodesic representative in  $[\alpha]$ . Moreover,  $D \rightarrow 0, C \rightarrow 1$  as  $\Lambda \rightarrow 0$ .

**1.3. The Thurston metric on  $\mathcal{T}_{g,n}(\Lambda)$ .** There is an asymmetric metric on  $\mathcal{T}_{g,n}(0)$  defined by

$$(2) \quad d_{\text{Th}}(X_1, X_2) = \log \sup_{[\alpha] \in \mathcal{S}(S)} \frac{l_{X_2}([\alpha])}{l_{X_1}([\alpha])},$$

which is the so called *Thurston metric* [28].

We combine Theorem A with a generalized Mcshane's identity on a hyperbolic cone surface due to Tam-Wong-Zhang [26] to show that

**Theorem C .** Let  $S$  be a non-exceptional surface. For any  $X_1, X_2 \in \mathcal{T}_{g,n}(\Lambda)$ , if  $l_{X_1}([\alpha]) \geq l_{X_2}([\alpha])$  for any  $[\alpha] \in \mathcal{S}(S)$ , then  $X_1 = X_2$ .

As a result, we can define the Thurston metric on  $\mathcal{T}_{g,n}(\Lambda)$  for any  $\Lambda \in (-\pi, \infty)^n$ , using the formula (2). It would be interesting to compare the Thurston metric with the arc metric on  $\mathcal{T}_{g,n}(\Lambda)$ , defined by Liu-Papadopoulos-Su-Th  ret [14].

Using Theorem B, we are able to prove that any two Teichm  ller spaces  $\mathcal{T}_{g,n}(\Lambda), \mathcal{T}_{g,n}(\Lambda')$  are almost isometric.

**Theorem D (Almost-isometry).** Let  $\Lambda \in (-\pi, \infty)^n$  and let  $S$  be a non-exceptional surface. The map  $F_{\Gamma, \mathcal{B}, \Lambda} : \mathcal{T}_{g,n}(\Lambda) \rightarrow \mathcal{T}_{g,n}(0)$  is an almost-isometry, i.e., there is a constant  $C$  depending on  $\Lambda$  such that

$$d_{\text{Th}}(X'_1, X'_2) - C \leq d_{\text{Th}}(X_1, X_2) \leq d_{\text{Th}}(X'_1, X'_2) + C, \text{ for any } X_1, X_2 \in \mathcal{T}_{g,n},$$

where  $X'_1 = F_{\Gamma}(X_1)$  and  $X'_2 = F_{\Gamma}(X_2)$ . Moreover,  $C \rightarrow 0$  as  $\Lambda \rightarrow 0$ .

**1.4. The Thurston boundary of  $\mathcal{T}_{g,n}(\Lambda)$ .** For the space  $\mathcal{T}_{g,n}(0)$ , the Thurston boundary is naturally identified with  $\mathcal{PML}(S)$ , the space of projective classes of measured laminations ([7]). We will prove that this is also true in our settings. Denote by  $\mathbb{R}_+^{\mathcal{S}}$  the set of non-negative functionals on the set of isotopy classes of non-peripheral simple closed curves (see Definition 2).

**Theorem E.** Suppose  $\Lambda \in (-\pi, \infty)^n$  and  $S$  is a non-exceptional surface. Let  $\Psi_\Lambda$  and  $\Pi$  be the maps defined as following:

$$\begin{aligned} \Psi_\Lambda : \mathcal{T}_{g,n}(\Lambda) &\longrightarrow \mathbb{R}_+^s \\ X &\longmapsto (l_X([\alpha]))_{\alpha \in \mathcal{S}(S)}, \end{aligned}$$

and

$$\begin{aligned} \Pi : \mathbb{R}_+^s &\longrightarrow P\mathbb{R}_+^s \\ (s_\alpha)_{\alpha \in \mathcal{S}(S)} &\longmapsto [(s_\alpha)_{\alpha \in \mathcal{S}(S)}]. \end{aligned}$$

Then

- (a) both  $\Psi_\Lambda$  and  $\Pi \circ \Psi_\Lambda$  are embeddings, where  $\mathcal{T}_{g,n}(\Lambda)$  is equipped with the topology induced by  $d_{\text{Th}}$  and  $\mathbb{R}_+^s$  is equipped with the weak topology;
- (b)  $\mathcal{T}_{g,n}(\Lambda) \ni X_n \rightarrow \xi \in \mathcal{PML}(S) \iff \mathcal{T}_{g,n}(0) \ni F_{\Gamma, \mathcal{B}, \Lambda}(X_n) \rightarrow \xi \in \mathcal{PML}(S)$ . As a result, the boundary of  $\mathcal{T}_{g,n}(\Lambda)$  in  $P\mathbb{R}_+^s$  (i.e., the Thurston boundary of  $\mathcal{T}_{g,n}(\Lambda)$ ) is homeomorphic to  $\mathcal{PML}(S)$ .

The paper is organized as following. In Section 2, we recall basic facts about hyperbolic cone surfaces. In Section 3, we study the Fenchel-Nielsen coordinates of the Teichmüller space  $\mathcal{T}_{g,n}(S)$ . In Section 4, we prove Theorem A and Theorem C. In Section 5, we compare the lengths of simple closed curves between hyperbolic cone surfaces and prove Theorem B. In Section 6, we prove Theorem D. In Section 7, we study some basic properties of the Teichmüller space  $\mathcal{T}_{g,n}(\Lambda)$  and prove Theorem E.

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## 2. HYPERBOLIC CONE METRICS

In this section, we study some basic properties of hyperbolic cone surfaces, with or without boundary, and with each cone angle less than  $\pi$ .

**Definition 1** ([4]). A *hyperbolic cone-surface* is a two-dimensional manifold  $X$ , with or without boundary, which can be triangulated by hyperbolic triangles.

The *singular locus*  $\text{Cone}(X)$  of a hyperbolic cone-surface  $X$  consists of interior point of  $X$  which have no neighbourhoods isometric to a

ball in the hyperbolic plane and boundary points which have no neighbourhoods isometric to a half ball in the hyperbolic plane. It follows that

- $Cone(X)$  is contained in the set of vertices of the hyperbolic triangulation of  $X$ , and it is a finite set;
- At each point of  $Cone(X)$ , there is a *cone angle* which is the sum of the angles of the dihedral angles of the triangles containing the point;
- $X \setminus Cone(X)$  has a smooth Riemannian metric of constant curvature  $-1$ , but this metric is incomplete if there is a cone point with positive cone angle;

In this paper, we are interested in the case where  $X$  satisfies the following requirements:

- (a)  $X$  has at most  $n$  cone points and every cone angle is strictly smaller than  $\pi$ . In particular, a cone point with zero cone angle is called a *cusp*.
- (b) Every boundary component of  $X$  consists of at most one geodesic.

**Remark 1.** To distinguish a positive cone angle point and the zero angle point, whenever we mention a cone point we mean a cone point with a positive cone angle.

**Definition 2.** A *generalized boundary component* of a cone surface  $X$  is a geodesic boundary component, a cusp, or a cone point. We denote by  $\Sigma_X$  the set of all generalized boundary components of  $X$ . A non-trivial simple closed curve on  $X$  is called *non-peripheral* if it is not homotopic to any generalized boundary component. A simple arc with endpoints on the generalized boundary components is called *non-peripheral* if it is not homotopic to any subarc of the generalized boundary components.

**Definition 3.** A *marked hyperbolic cone surface* is a pair  $(f, X)$ , where  $X$  is a hyperbolic cone-surface, and  $f : S \rightarrow X$  is a homeomorphism. Two marked hyperbolic cone-surfaces  $(f, X)$  and  $(f', X')$  are called equivalent if there is an isometry isotopic to  $f \circ (f')^{-1}$ .

A necessary condition of two marked hyperbolic cone surfaces to be equivalent is that they have the same numbers of cone points, of cusps and of geodesic boundaries. Denote by  $\mathcal{T}_{g,n}(S)$  the space of equivalent classes of marked hyperbolic cone surfaces.

**Remark 2.** Since  $S$  is an oriented surface obtained by removing  $n$  points from a closed surface, a marked hyperbolic cone surface may induce an incomplete metric of curvature  $-1$  on  $S$ . From now on,

whenever we talk about a hyperbolic cone metric on  $S$ , we mean its metric completion.

Next, we collect some basic properties of hyperbolic cone surfaces, for more details, we refer to [4], [5] and [26].

**Proposition 2.1** ([5][26]). *Let  $X$  be a non-exceptional cone-surface.*

- (a) *Every non-trivial simple closed curve on  $X \setminus \Sigma_X$  is freely homotopic to either a unique simple closed geodesic or a unique cone point or a cusp.*
- (b) *If two distinguished, non-trivial closed curves  $\alpha$  and  $\beta$  intersect  $n$  times, their corresponding geodesic will intersect at most  $n$  times.*
- (c) *Given two non-intersecting smooth simple curves  $\alpha$  and  $\beta$  on  $S$  there is at least one geodesic path  $c$  between them such that  $d(\alpha, \beta)$  is realized by  $c$ . Such a path  $c$  is perpendicular to  $\alpha$  and  $\beta$ . If  $\alpha$  and  $\beta$  are geodesic, in a free homotopy class of paths with end points moving on  $\alpha$  and  $\beta$ , such a path  $c$  is unique. This property remains true for singular points in place of one or both geodesics.*

**Theorem 2.2** (Pants decomposition [5]). *Let  $X$  be a non-exceptional cone-surface of genus  $g$  with  $n$  cone points  $\Delta_1, \dots, \Delta_n$ . Let  $\gamma_1, \dots, \gamma_m$  be disjoint simple closed geodesics on  $M$ . Then the followings hold.*

- (a)  $m \leq 3g - 3 + n$ .
- (b) *There exist simple closed geodesics  $\gamma_{m+1}, \dots, \gamma_{3g-3+n}$  which together with  $\gamma_1, \dots, \gamma_m$  form a partition of  $S$ .*

Denote by  $[\alpha]$  the isotopy class of a simple closed curve  $\alpha$ . Denote by  $\mathcal{S}(S)$  the set of isotopy classes of non-peripheral simple closed curves. From Proposition 2.1, it follows that every hyperbolic cone metric on  $S$  induces a functional on  $\mathcal{S}(S)$  defined by

$$\begin{aligned} l : \mathcal{T}_{g,n} &\longrightarrow \mathbb{R}_+^{\mathcal{S}} \\ X &\longmapsto (l_X([\alpha]))_{[\alpha] \in \mathcal{S}(S)}, \end{aligned}$$

where  $l_X([\alpha])$  represents the length of the geodesic representative in  $[\alpha]$ , and where  $\mathbb{R}_+^{\mathcal{S}}$  is the space of nonnegative functionals on  $\mathcal{S}(S)$ . We equip  $\mathbb{R}_+^{\mathcal{S}}$  with the weak topology.

**Definition 4.** The sequence  $(l_X([\alpha]))_{[\alpha] \in \mathcal{S}(S)}$  is called *the marked length spectrum* about non-peripheral simple closed curves of  $X$ , denoted by  $\mathcal{MLSS}(X)$ .

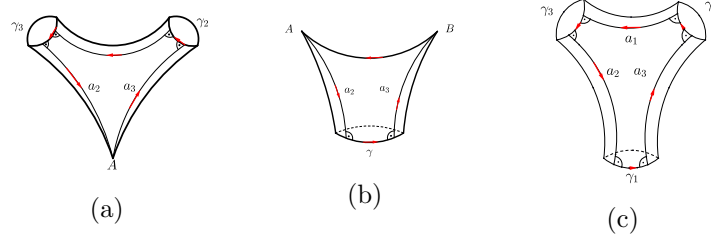


FIGURE 1. Generalized Y-pieces. They are V-piece, Joker's hat and Y-piece respectively.

**2.1. Generalized Y-pieces.** A *Y-piece* is a sphere with three geodesic boundary components; a *V-piece* is a sphere with two geodesic boundary components and a cone point with cone angle less than  $\pi$ , or a sphere with two geodesic boundary components and a cusp; and a *Joker's hat* is a sphere with a geodesic boundary component and two cone points with each cone angle less than  $\pi$ , or a sphere with a geodesic boundary component and two cusps, or a sphere with a geodesic boundary component, a cone point with cone angle less than  $\pi$  and a cusp. For convenience, all these pieces are called *generalized Y-pieces*, or pairs of pants (see Fig. 1).

**2.2. Generalized X-piece.** A generalized  $\mathcal{X}$ -piece is a hyperbolic cone surface obtained by pasting together two generalized Y-pieces along two boundary geodesics of the same length. Let  $G$  be a V-piece with generalized boundary components  $\gamma_1, \gamma_2, \gamma_3$ . Let  $G'$  be a Joker's hat with generalized boundary components  $\gamma'_1, \gamma'_2, \gamma'_3$ . Assume that  $\gamma_1, \gamma'_1$  are geodesic boundary components and that they have the same length  $l$ . Recall that a generalized Y-piece consists of two isometric hyperbolic polygons. Choose an orientation of  $\gamma_1$  (resp.  $\gamma'_1$ ) such that  $G$  (resp.  $G'$ ) sits on the left. Parameterize  $\gamma_1$  (resp.  $\gamma'_1$ ) by arc length such that the basepoint '0' is one of the two vertices of the corresponding polygon contained in  $\gamma_1$  (resp.  $\gamma'_1$ ). We paste  $G$  and  $G'$  along  $\gamma_1, \gamma'_1$  with pasting condition:

$$(3) \quad \gamma_1(s) = \gamma'_1(tl - s), \quad s \in \mathbb{R}/\sim, \quad t \in \mathbb{R},$$

where  $\sim$  represents an equivalent relation such that  $s \sim s'$  if  $s - s' = kl$  for some  $k \in \mathbb{Z}$ . In this way, we get a *VJ-piece*  $X^t$ :

$$X^t \triangleq G \cup G' \quad \text{mod } (3).$$

The curve  $\gamma_1$  ( $\gamma'_1$ ) is called the *waist* of the *VJ piece*.

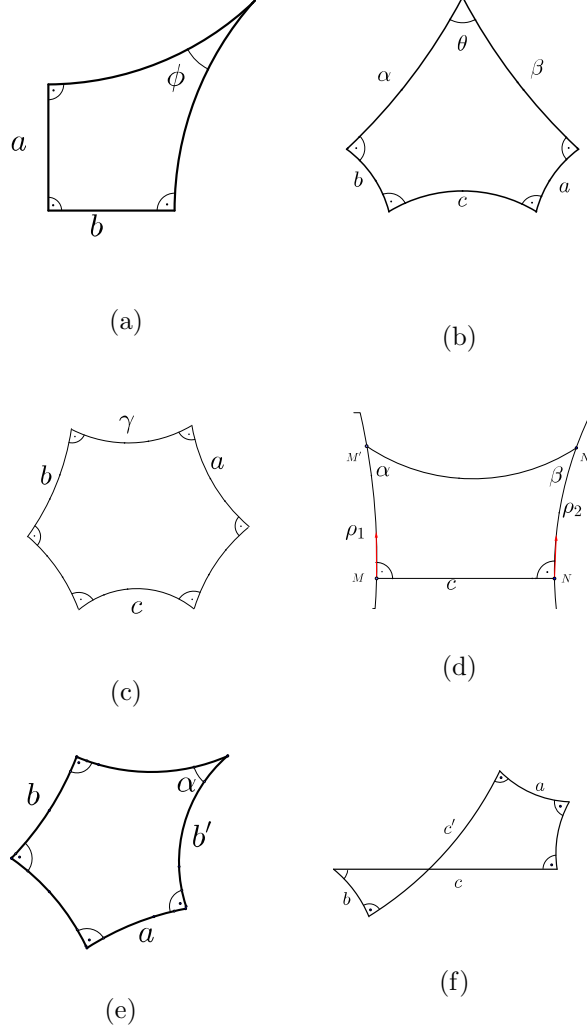


FIGURE 2. Hyperbolic polygons

By similar operations, we get *YY-piece*, *YV-piece*, *YJ-piece*, *VV-piece*, *JJ-piece*. All these pieces are called *generalized  $\mathcal{X}$ -pieces*.

It follows from the pasting condition (3) that  $X^{t+1}$  is isometric to  $X^t$ , i.e.  $t$  is defined only in  $\mathbb{R}/\mathbb{Z}$ . To extend the domain of  $t$  from  $\mathbb{R}/\mathbb{Z}$  to  $\mathbb{R}$ , we need to add some “marking” to  $X^t$  (see §3).

**2.3. Hyperbolic geometry.** For convenience, we collect some identities of hyperbolic geometry which can be found in [1] and [8].

**Lemma 2.3** ([1],[8]). *Elementary formulae of hyperbolic geometry:*



- *Tri-rectangle, see Fig. 2(a),*

$$(4) \quad \cos \phi = \sinh a \sinh b.$$

- *Pentagon, see Fig. 2(b),*

$$(5) \quad \cosh c = -\cosh \alpha \cosh \beta \cos \theta + \sinh \alpha \sinh \beta,$$

$$(6) \quad \cos \theta = \sinh a \sinh b \cosh c - \cosh a \cosh b.$$

- *Hexagon, see Fig. 2(c),*

$$(7) \quad \cosh c = \sinh a \sinh b \cosh \gamma - \cosh a \cosh b.$$

- *Quadrilaterals with two right angles, see 2(d),*

$$(8) \quad \cosh d(M', N') = \cosh \rho_1 \cosh \rho_2 \cosh c - \sinh \rho_1 \sinh \rho_2,$$

where  $\rho_1$  (resp.  $\rho_2$ ) represents oriented distance from  $M$  to  $M'$  (resp. from  $N$  to  $N'$ ). If  $\rho_1 \rho_2 > 0$ , then

$$(9) \quad \cos \alpha = -\cos \beta \cosh c + \sin \beta \sinh c \sinh |\rho_2|.$$

and

$$(10) \quad \cosh c = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cosh d(M', N').$$

- *Convex pentagon with four right angles, see Fig. 2(e)*

$$(11) \quad \cosh b = \sin \alpha \sinh a \sinh b' - \cos \alpha \cosh a.$$

- *The self-intersecting pentagon with four right angles, see Fig. 2(f)*

$$(12) \quad \sinh c = \sinh \alpha \cosh b \cosh c' + \cosh a \sinh b.$$

### 3. THE FENCHEL-NIELSEN COORDINATES

In this section, we study the Fenchel-Nielsen coordinates for the Teichmüller space  $\mathcal{T}_{g,n}$ . First of all, we need to choose a *coordinate system of curves* on  $S_{g,n}$  which consists of the following data:

- a *pants decomposition* which is a set of non-peripheral, oriented simple closed curves  $\Gamma = \{\gamma_1, \dots, \gamma_{3g-3+n}\}$  such that  $[\gamma_i] \neq [\gamma_j]$  if  $i \neq j$  and  $S \setminus \Gamma$  consists of pairs of pants  $\mathcal{R} = \{R_i\}$ ,  $i = 1, 2, \dots, 2g-2+n$  (see Fig. 3);
- a set of *seams*  $\mathcal{B} = \{\beta_1, \dots, \beta_k\}$  which is a collection of disjoint non-peripheral simple closed curves or non-peripheral simple arcs such that the intersection of the union  $\cup_{j=1}^k \beta_j$  with any pair of pants  $R \in \{R_1, \dots, R_{2g-2+n}\}$  determined by the pants decomposition  $\Gamma$  is a union of three disjoint arcs connecting the boundary components of  $R$  pairwise.

Let  $R$  be a pair of pants with three oriented boundary components  $\gamma_1, \gamma_2, \gamma_3$ , and  $\delta$  be the geodesic perpendicular to both  $\gamma_1$  and  $\gamma_2$ . Let  $\beta$  be a simple arc on  $R$  connecting  $\gamma_1$  and  $\gamma_2$ . There exists a homotopy  $H$  between  $\delta$  and  $\beta$  which keeps the endpoints on the boundary of  $R$ . The *twisting number* of  $\beta$  at  $\gamma_1$  is defined to be the signed displacement from  $\delta \cap \gamma_1$  to  $\beta \cap \gamma_1$  during the homotopy. The twisting number of  $\beta$  at  $\gamma_2$  is defined similarly.

Now we define the *twist parameter*  $t_i$  of  $X \in \mathcal{T}_{g,n}$  along  $\gamma_i \in \Gamma$ . Let  $\beta_j \in \mathcal{B}$  be one of the two seams crossing  $\gamma_i$ . On each side of the geodesic representative of  $\gamma_i$  there is a pair of pants, and the geodesic representative of  $\beta_j$  gives an geodesic arc on each of these two pairs of pants. Let  $t_{il}$  and  $t_{ir}$  be the twisting numbers of each of these arcs on the left and right side of the geodesic representative of  $\gamma_i$ , respectively. The *twist parameter* of  $X$  along  $\gamma_i$  is defined to be:

$$t_i := \frac{t_{il} - t_{ir}}{l_X(\gamma_i)},$$

where  $l_X(\gamma_i)$  represents the length of the geodesic representative of  $\gamma_i$  on  $X$ .

**Remark 3.** (1) The sign of the twist parameter of  $X$  along  $\gamma_i$  does not depend on the orientation of  $\gamma_i$ . Instead, it depends on the orientation of the surface.

(2) The twist parameters depend on the choice of seams  $\mathcal{B}$ . Let  $\mathcal{B}'$  be another set of seams and  $\{t'_i\}$  be the corresponding twist parameters. Then  $t_i - t'_i \in \mathbb{Z}/2$ . Moreover, the difference  $t_i - t'_i$  is independent of  $X$ , it depends only on the seams  $\mathcal{B}$  and  $\mathcal{B}'$ .

Now we fix a *coordinate system of curves*  $(\Gamma, \mathcal{B})$  and the corresponding pairs of pants  $\mathcal{R}$  for  $S_{g,n}$ . We construct a marked hyperbolic cone surface for any given  $(6g - 6 + 3n)$ -tuple  $(\Lambda, L, T) \in (-\pi, \infty)^n \times \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}$  as follows:

- $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  is the boundary assignments for the generalized boundary components  $\Delta_1, \dots, \Delta_n$ ;
- $L = (l([\gamma_1]), l([\gamma_2]), \dots, l([\gamma_{3g-3+n}]))$  is the lengths of the pants curves  
 $[\gamma_1], \dots, [\gamma_{3g-3+n}] \in \Gamma$ ;
- $T = (t_1, t_2, \dots, t_{3g-3+n})$  is the twist parameters along  $\gamma_1, \dots, \gamma_{3g-3+n} \in \Gamma$ .

$\Lambda$  and  $L$  determine the geometry of each pair of pants  $R_i \in \mathcal{R}$  and  $T$  determines how to paste them. For each  $\gamma_i \in \Gamma$ , there are two pairs of pants  $R, R' \in \mathcal{R}$  ( $R$  and  $R'$  may be the same pair of pants) such that  $\gamma_i$  serves as a common boundary component of them. Then  $t_i \in T$

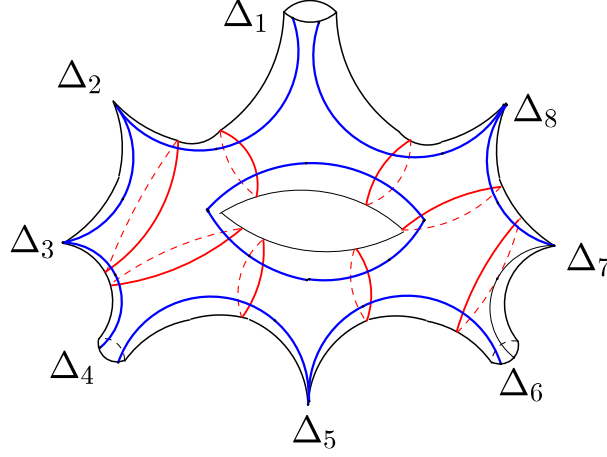


FIGURE 3. The hyperbolic cone surface  $X$  in this figure has one genus and eight generalized boundary components: three geodesic boundaries ( $\Delta_1, \Delta_4, \Delta_6$ ), four cone points ( $\Delta_2, \Delta_3, \Delta_7, \Delta_8$ ) and one cusp ( $\Delta_5$ ). The pants curves (the red circles) decompose  $X$  into eight generalized Y-pieces. The set consisting of blue arcs and blue circle consist a set of seams.

determines how  $R$  and  $R'$  are pasted (for more details about Fenchel-Nielsen coordinates, we refer to [6, §10.6]).

The construction above actually defines a map  $\Phi_{\Gamma, \mathcal{B}}$  as below:

$$(13) \quad \begin{array}{ccc} \Phi_{\Gamma, \mathcal{B}} : & (-\pi, \infty)^n \times \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n} & \longrightarrow \mathcal{T}_{g,n}(S) \\ & (\Lambda, L, T) & \longmapsto X. \end{array}$$

In the case that  $S$  is a closed surface,  $\Phi_{\Gamma}$  is a bijection (also homeomorphism). We will show, by Theorem 4.1 in § 4.1, that this is also true in our case. The  $6g - 6 + 3n$ -tuple  $(\Lambda, L, T)$  is called the Fenchel-Nielsen coordinates of  $X$  with respect to  $(\Gamma, \mathcal{B})$ , and the map  $\Phi_{\Gamma, \mathcal{B}}$  is called the Fenchel-Nielsen parametrization of  $\mathcal{T}_{g,n}(S)$  with respect to  $\Gamma$  and  $\mathcal{B}$ .

### 3.1. Boundary assignments and twist of a generalized X-piece.

First, we consider the boundary assignments of a generalized  $\mathcal{X}$ -piece.

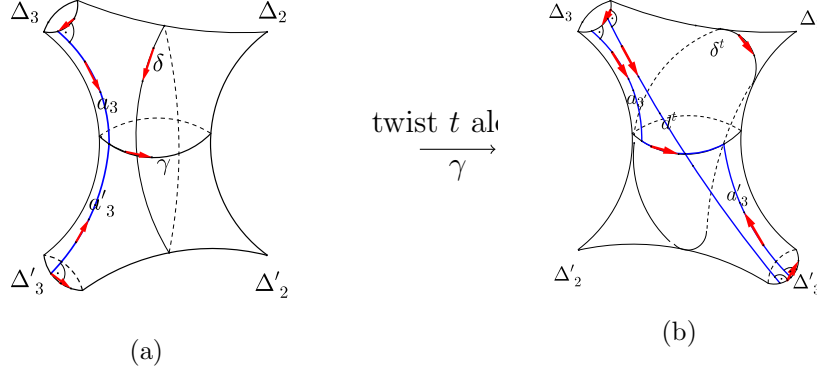


FIGURE 4. Constructions of generalized  $\mathcal{X}$ -pieces by gluing two pairs of pants along two chosen boundaries and twisting to the left about length  $t|\gamma|$ . Orientations of curves:  $X$  sits on the left of  $\Delta_3$  and  $\Delta'_3$ , the cone points  $\Delta_2, \Delta'_2$  sit on the left of  $\delta$ ,  $\Delta_2, \Delta_3$  sit on the left of  $\gamma$ , the orientation of  $a_3$  (resp.  $a'_3$ ) is chosen from  $\Delta_3$  (resp.  $\Delta'_3$ ) to  $\gamma$ .

**Lemma 3.1** (Boundary assignments). *Let  $X$  be a generalized  $\mathcal{X}$ -piece with generalized boundary components  $\Delta_2, \Delta'_2, \Delta_3, \Delta'_3$ . If  $\Delta_3, \Delta'_3$  are geodesic boundaries with given lengths  $\lambda_3, \lambda'_3$ , then the boundary assignments of  $\Delta'_2, \Delta_2$  are determined by  $\lambda_3, \lambda'_3$  and  $\mathcal{MLSS}(X)$ . Moreover, we need at most 28 fixed non-peripheral simple closed curves.*

*Proof.* A generalized boundary can be a cone point, a cusp, or a closed geodesic. First, we consider the case that the  $\Delta_2, \Delta'_2$  are cone points or cusps (see Fig. 4(a)).

Let  $\gamma$  be a non-peripheral simple closed curve of  $\mathcal{X}$  and  $\mathcal{B} = \{\beta_1, \beta'_1, \beta_2, \beta'_2\}$  a collection of seams, where  $\beta_1$  (resp.  $\beta'_1$ ) connects  $\Delta_2$  and  $\Delta_3$  (resp.  $\Delta'_2$  and  $\Delta'_3$ ) and  $\beta_2$  (resp.  $\beta'_2$ ) connects  $\Delta_3$  and  $\Delta'_3$  (resp.  $\Delta_2$  and  $\Delta'_2$ ). Let  $\delta$  be the simple closed curve homotopic to  $\beta_2 \cdot \gamma'_3 \cdot (\beta_2)^{-1} \cdot \gamma_3$ . The waist  $\gamma$  cuts  $X$  into two pairs of pants with the third boundaries  $\Delta_1, \Delta'_1$ , respectively. Let  $X_t$  be the generalized  $\mathcal{X}$ -piece obtained by gluing these two pairs of pants along  $\Delta_1, \Delta'_1$  with twist amount  $t$  with respect to  $\mathcal{B}$  (see Fig. 4(b)).

Denote by  $Tw_\gamma^n$  the  $n$  times Dehn twist along  $\gamma$ . Let  $\delta_n$  be the simple closed curve obtained from  $\delta$  by  $n$  times Dehn twist along  $\gamma$ , i.e.  $\delta_n = Tw_\gamma^n \delta$ . Denote by  $a_3$  (resp.  $a'_3$ ) the geodesic perpendicular to both  $\Delta_3$  and  $\Delta_1$  (resp.  $\Delta'_3$  and  $\Delta'_1$ ). All the directions (except

for  $\Delta_1, \Delta'_1$ ) are illustrated in Fig. 4. Let  $d^t$  and  $\delta^t$  be the geodesic representatives of  $\beta_2$  and  $\delta$  on  $X_t$ , respectively.

With the notations above, we have the following formula which can be found in [1, Prop.3.3.11]

$$(14) \quad \cosh \frac{1}{2} |\delta_n| = \sinh(\frac{1}{2} \lambda_3) \sinh(\frac{1}{2} \lambda'_3) \{ \sinh |a_3| \sinh |a'_3| \cosh(t+n)|\gamma| + \cosh |a_3| \cosh |a'_3| \} - \cosh(\frac{1}{2} \lambda_3) \cosh(\frac{1}{2} \lambda'_3),$$

where  $|\delta_n|, |a_3|, |a'_3|, |\gamma|, \lambda_3, \lambda'_3$  represent the lengths of the corresponding geodesic representatives.

Hence

$$(15) \quad \cosh \frac{|\delta_1|}{2} - \cosh \frac{|\delta|}{2} = \sinh \frac{\lambda_3}{2} \sinh \frac{\lambda'_3}{2} \sinh |a_3| \sinh |a'_3| (\cosh(t|\gamma| + |\gamma|) - \cosh(t|\gamma|)),$$

and

$$(16) \quad \frac{\cosh \frac{|\delta_2|}{2} - \cosh \frac{|\delta_1|}{2}}{\cosh \frac{|\delta_1|}{2} - \cosh \frac{|\delta|}{2}} = \frac{\cosh(t|\gamma| + 2|\gamma|) - \cosh(t|\gamma| + |\gamma|)}{\cosh(t|\gamma| + |\gamma|) - \cosh(t|\gamma|)}.$$

From this we know that  $t$  is determined by  $|\delta|, |\delta_1|, |\delta_2|$  and  $|\gamma|$ .

Squaring (6) and rearranging, we have

$$(17) \quad \sinh^2 \frac{\lambda_3}{2} \sinh^2 |a_3| = \sinh^{-2} \frac{|\gamma|}{2} [\cosh^2 \frac{|\gamma|}{2} + 2 \cos \frac{\lambda_2}{2} \cosh \frac{\lambda_3}{2} \cosh \frac{|\gamma|}{2} + \cos^2 \frac{\lambda_2}{2} + \cosh^2 \frac{\lambda_3}{2} - 1],$$

where  $-\lambda_2$  is the cone angle at the cone point  $\Delta_2$ .

Set

$$B_2 \triangleq 2 \cos \frac{\lambda_2}{2} \cosh \frac{\lambda_3}{2}, \quad C_2 \triangleq \cos^2 \frac{\lambda_2}{2} + \cosh^2 \frac{\lambda_3}{2} - 1; \\ B'_2 \triangleq 2 \cos \frac{\lambda'_2}{2} \cosh \frac{\lambda'_3}{2}, \quad C'_2 \triangleq \cos^2 \frac{\lambda'_2}{2} + \cosh^2 \frac{\lambda'_3}{2} - 1.$$

Then

$$(18) = \frac{(\cosh \frac{|\delta_1|}{2} - \cosh \frac{|\delta|}{2})^2 \sinh^4 \frac{|\gamma|}{2}}{[\cosh(t|\gamma| + |\gamma|) - \cosh(t|\gamma|)]^2} \\ + \cosh^4 \frac{|\gamma|}{2} + (B_2 + B'_2) \cosh^3 \frac{|\gamma|}{2} + (C_2 + C'_2 + B_2 B'_2) \cosh^2 \frac{|\gamma|}{2} \\ + (B_2 C'_2 + B'_2 C_2) \cosh \frac{|\gamma|}{2} + C_2 C'_2.$$

Equation (18) is a linear equation about parameters  $B_2 + B'_2, C_2 + C'_2 + B_2 B'_2, B_2 C'_2 + B'_2 C_2$  and  $C_2 C'_2$ .

Now, changing the pants decomposition curve from  $\gamma$  to  $\gamma_k = Tw_\delta^k \gamma$ ,  $k = \pm 1, \pm 2, \pm 3$ , where  $Tw_\delta^n$  represents the  $n$  times Dehn twist along  $\delta$ , we get six more linear equations about parameters  $B_2 + B'_2$ ,  $C_2 + C'_2 + B_2 B'_2$ ,  $B_2 C'_2 + B'_2 C_2$  and  $C_2 C'_2$ .

Combining (11) and (12), we have the following formula

$$(19) \quad \cosh \frac{1}{2} |\gamma_k| = \sinh(\frac{1}{2} \lambda_3) \sin(\frac{1}{2} \lambda_2) \{ \sinh |b_3| \cosh |b_2| \cosh(\tilde{t} + k) |\delta| + \cosh |b_3| \sinh |b_2| \} - \cos(\frac{1}{2} \lambda_2) \cosh(\frac{1}{2} \lambda_3),$$

where  $b_3$  (resp.  $b_2$ ) is the length of the geodesic perpendicular to both  $\Delta_3$  and  $\delta$  (resp.  $\Delta_2$  and  $\delta$ ),  $\tilde{t}$  represents the twist of  $X$  along  $\delta$ .

It follows from (29) that if  $\tilde{t} < 0$ , then  $|\gamma_{-3}|$ ,  $|\gamma_{-2}|$ ,  $|\gamma_{-1}|$  and  $|\gamma_0|$  are pairwise different, if  $\tilde{t} \geq 0$ , then  $|\gamma_3|$ ,  $|\gamma_2|$ ,  $|\gamma_1|$  and  $|\gamma_0|$  are pairwise different. Hence, at least one of the following two matrices has non-zero determinant,

$$\begin{pmatrix} \cosh^3 \frac{\gamma_3}{2} & \cosh^2 \frac{\gamma_3}{2} & \cosh \frac{\gamma_3}{2} & 1 \\ \cosh^3 \frac{\gamma_2}{2} & \cosh^2 \frac{\gamma_2}{2} & \cosh \frac{\gamma_2}{2} & 1 \\ \cosh^3 \frac{\gamma_1}{2} & \cosh^2 \frac{\gamma_1}{2} & \cosh \frac{\gamma_1}{2} & 1 \\ \cosh^3 \frac{\gamma_0}{2} & \cosh^2 \frac{\gamma_0}{2} & \cosh \frac{\gamma_0}{2} & 1 \end{pmatrix}, \quad \begin{pmatrix} \cosh^3 \frac{\gamma_{-3}}{2} & \cosh^2 \frac{\gamma_{-3}}{2} & \cosh \frac{\gamma_{-3}}{2} & 1 \\ \cosh^3 \frac{\gamma_{-2}}{2} & \cosh^2 \frac{\gamma_{-2}}{2} & \cosh \frac{\gamma_{-2}}{2} & 1 \\ \cosh^3 \frac{\gamma_{-1}}{2} & \cosh^2 \frac{\gamma_{-1}}{2} & \cosh \frac{\gamma_{-1}}{2} & 1 \\ \cosh^3 \frac{\gamma_0}{2} & \cosh^2 \frac{\gamma_0}{2} & \cosh \frac{\gamma_0}{2} & 1 \end{pmatrix}.$$

It follows that there is a unique solution for the 4-tuple  $(B_2 + B'_2 C_2 + C'_2 + B_2 B'_2, B_2 C'_2 + B'_2 C_2, C_2 C'_2)$ . Further, the boundary assignments  $\lambda_2, \lambda'_2$  can be uniquely obtained from  $B_2 + B'_2$  and  $C_2 C'_2$ .

If  $\delta_2$  (resp.  $\delta'_2$ ) is a geodesic, we need to find the corresponding equations similar to (16) and (17). Repeating the calculations above using (11) and (12), we find that the equation corresponding to (16) is exactly the same as (16), while the equation corresponding to (17) is a little bit different from (17) that  $\cos \frac{\lambda_2}{2}$  is replaced by  $\cosh \frac{\lambda_2}{2}$ .

It follows from equation (16) that, for each one of the seven curves  $\{Tw_\delta^i \gamma : i = 0, \pm 1, \pm 2, \pm 3\}$ , four curves (including  $Tw_\delta^i$  itself) are involved to determine the twist parameter. Therefore, we need at most 28 curves.

□

The proof of Lemma 3.1 also proves that the twist amount of a generalized  $\mathcal{X}$ -piece with respect to a waist and a collection of seams is determined by its marked length spectrum.

**Lemma 3.2** (Twist). *Let  $X$  be a generalized  $\mathcal{X}$ -piece with generalized boundary components  $\Delta_2, \Delta'_2, \Delta_3, \Delta'_3$ , and with boundary assignments  $\Lambda = (\lambda_2, \lambda'_2, \lambda_3, \lambda'_3)$ . Let  $(\Gamma, \mathcal{B})$  be a coordinate system of curves of  $X$ . Then the twist parameter  $t$  is determined by  $\Lambda$  and  $\mathcal{MLSS}(X)$ . Moreover, we need at most 4 non-peripheral simple closed curves.*

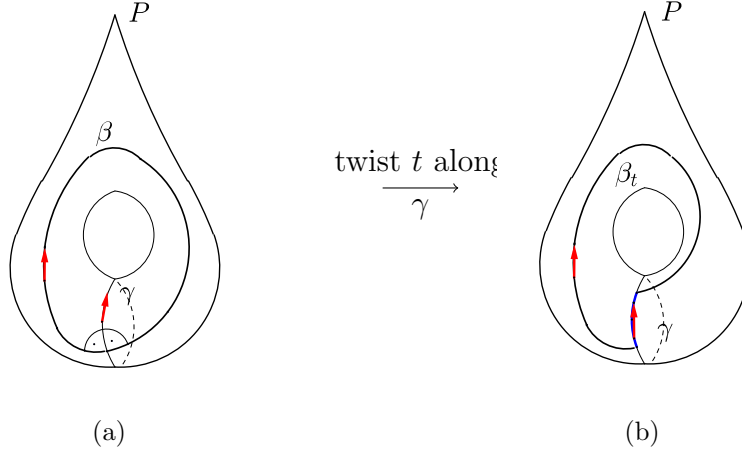


FIGURE 5. Figure (a) is a torus with one cone point and with no twist along  $\gamma$  with respect to  $\beta$ , figure (b) is a torus with one cone point and with twist  $t$  along  $\gamma$  with respect to  $\beta$ .

*Proof.* It follows immediately from (16). □

#### 4. MARKED LENGTH SPECTRAL RIGIDITY OVER $\mathcal{T}_{g,n}$

##### 4.1. Marked length spectral rigidity about simple closed curves.

In this section we investigate the relationship between the geometry of a hyperbolic cone surface and its marked length spectrum.

**Theorem 4.1.** *Let  $S$  be a non-exceptional surface.  $\mathcal{S}(S)$  is rigid over  $\mathcal{T}_{g,n}(S)$ . More precisely, let  $X_1, X_2 \in \mathcal{T}_{g,n}(S)$ , if  $l_{X_1}([\alpha]) = l_{X_2}([\alpha])$  for any  $[\alpha] \in \mathcal{S}(S)$ , then  $X_1 = X_2$ .*

*Proof.* We distinguish two cases.

**Case 1.**  $S$  is a torus with only one generalized boundary component. Let  $X$  be such a hyperbolic cone surface (see Fig. 5(a) and Fig. 5(b) for the case that the generalized boundary is a cone point). We prove for the case where the generalized boundary is a cone point. The other cases are similar. First, we choose a non-peripheral simple closed curve  $\gamma$  and a collection of seams  $\mathcal{B} = \{\beta, \beta'\}$  where  $\beta$  is a non-peripheral simple closed curve which intersects  $\gamma$  only once (see Fig. 5(a)) and  $\beta'$  is a simple arc with endpoints on the generalized boundary component.

Then the curve system  $(\{\gamma\}, \mathcal{B})$  defines a Fenchel-Nielsen coordinates of  $X$ . Let  $X_t$  be the hyperbolic cone surface whose Fenchel-Nielsen coordinates are  $(\lambda(X), l_X(\gamma), t)$ , where  $\lambda(X)$  is the cone angle of  $X$  and  $l_X(\gamma)$  is the length of the geodesic representative of  $\gamma$  on  $X$ .

To prove the theorem, we need to show that the cone angle and the twist are determined by the lengths of non-peripheral curves. Let  $\beta_n$  be the curve obtained from  $\beta$  by  $n$  times Dehn twist along  $\gamma$ . Then the twist of  $X$  along  $\gamma$  with respect to  $\beta_n$  is  $t + n$ . Let  $t_{n,l}$  and  $t_{n,r}$  be the twist numbers of (geodesic representative of)  $\beta_n$  on the left and right side of (the geodesic representative of)  $\gamma$ . It follows from the definition of twist parameter that  $t_{n,l} - t_{n,r} = (t + n)l_X(\gamma)$ .

Cutting  $X$  along  $\gamma$ , we get a pair of pants  $R$  where  $\gamma$  corresponds to two geodesic boundary components. By the definition of twist parameter, the length  $l_{X_0}(\beta)$  of  $\beta$  on  $X_0$  coincides with the shortest distance between these two geodesic boundary components. As a consequence, the cone angle  $\lambda(X)$  of  $X$  is determined by  $l_{X_0}(\beta)$  and  $l_X(\gamma)$ . More precisely, it follows from (6) that

$$(20) \quad \cos \frac{\lambda(X)}{2} = \sinh^2 \frac{l_X(\gamma)}{2} \cosh l_{X_0}(\beta) - \cosh^2 \frac{l_X(\gamma)}{2}.$$

Combining the definition of twist parameter and Equation (8), the length  $l_X(\beta_n)$  of  $\beta_n$  on  $X$  is

$$l_X(\beta_n) = \cosh^{-1}(\cosh t_{n,l} \cosh t_{n,r} \cosh l_{X_0}(\beta) - \sinh t_{n,l} \sinh t_{n,r}).$$

Hence

$$\begin{aligned} & \cosh l_X(\beta_n) \\ &= \inf_{s \in \mathbb{R}} \cosh s \cosh [(s - t - n)l_X(\gamma)] \cosh l_{X_0}(\beta) - \sinh s \sinh [(s - t - n)l_X(\gamma)] \\ &= 2 \cosh \frac{(t+n)l_X(\gamma)}{2} \cosh \frac{l_{X_0}(\beta)}{2} - 1. \end{aligned}$$

Therefore,

$$(21) \quad \cosh \frac{l_X(\beta_n)}{2} = \cosh \frac{(t+n)l_X(\gamma)}{2} \cosh \frac{l_{X_0}(\beta)}{2}.$$

Then

$$(22) \quad \frac{\cosh \frac{l_X(\beta_2)}{2} - \cosh \frac{l_X(\beta_1)}{2}}{\cosh \frac{l_X(\beta_1)}{2} - \cosh \frac{l_X(\beta)}{2}} = \frac{\sinh \frac{(2t+3)l_X(\gamma)}{4}}{\sinh \frac{(2t+1)l_X(\gamma)}{4}}.$$

From (20), (21) and (22), the twist parameter  $t$  and the cone angle  $\lambda(X)$  can be determined uniquely by  $l_X(\beta_2)$ ,  $l_X(\beta_1)$ ,  $l_X(\beta)$ ,  $l_X(\gamma)$ .

**Case 2.**  $S$  is not a torus with only one generalized boundary component. In this case, for each generalized boundary component  $\Delta_i$ , we can find at least one generalized  $\mathcal{X}$ -piece  $\mathcal{X}_i$  such that at least two



of the generalized boundary components of  $\mathfrak{X}_i$  are non-peripheral simple closed curves of  $S$ . It follows from Lemma 3.1 that the boundary assignments  $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  of  $X_1, X_2$  are the same.

Next, we fix a pants decomposition  $\Gamma = \{\gamma_i\}_{i=1}^{3g-3+n}$  and a collection of seams  $\mathcal{B} = \{\beta_1, \dots, \beta_k\}$ . To prove the theorem, it suffices to show that if  $l_{X_1}([\alpha]) = l_{X_2}([\alpha])$  for every  $[\alpha] \in \mathcal{S}(S)$  then  $X_1$  and  $X_2$  have the same Fenchel-Nielsen coordinates with respect to  $(\Gamma, \mathcal{B})$ . The pants lengths  $L = (l(\gamma_1), l(\gamma_2), \dots, l(\gamma_{3g-3+n}))$  can be uniquely determined, and the boundary assignments  $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  can also be uniquely determined as above. Finally, by Lemma 3.2, the twist parameters  $\mathcal{T} = (t_1, t_2, \dots, t_{3g-3+n})$  can also be uniquely determined. This completes the proof.  $\square$

In fact, the set  $\mathcal{S}(S)$  is unnecessarily large for the rigidity of  $\mathcal{T}_{g,n}(S)$  in the case where  $S$  is a non-exceptional surface. From the proof of Lemma 3.2 and Lemma 3.1, we know that at most  $28n + 4(3g - 3 + n)$  non-peripheral simple closed curves are involved. We summarize this as following:

**Theorem 4.2.** *There is a subset  $\mathcal{S}'(S) \subset \mathcal{S}(S)$  consisting of at most  $12g - 12 + 32n$  elements such that  $\mathcal{S}'(S)$  is rigid over  $\mathcal{T}_{g,n}(S)$  provided that  $S$  is a non-exceptional surface.*

Now Theorem A follows from Theorem 4.1 and Theorem 4.2.

**Remark 4.** In the case of closed surface, the subset  $\mathcal{S}'(S)$  can be chosen such that it has  $6g - 5$  elements [22].

If we consider the subspace  $\mathcal{T}_{g,n}(\Lambda)$  of  $\mathcal{T}_{g,n}$ , we get the following result.

**Theorem 4.3.** *Suppose  $\Lambda \in (-\pi, \infty)^n$  and that  $S$  is a non-exceptional surface. Then  $\mathcal{S}(S)$  is rigid over  $\mathcal{T}_{g,n}(\Lambda)$ , i.e. if  $l_{X_1}([\alpha]) = l_{X_2}([\alpha])$  holds for any  $[\alpha] \in \mathcal{S}(S)$ , then  $X_1 = X_2$ .*

*Proof.* The proof is similar to that of Theorem 4.1 except that we do not need to deal with the boundary assignments here.  $\square$

**4.2. Generalized marked length spectral rigidity.** In [26], the authors proved a generalized McShane's identity for length series of simple closed geodesics on a hyperbolic cone surface by studying "gaps" formed by simple-normal geodesics emanating from a distinguished cone point, cusp or boundary geodesic. For each pair of pants with

three generalized boundary components  $\Delta_0, \alpha, \beta$ , where  $\Delta_0$  is the distinguished generalized boundary component, they define a *Gap function*  $\text{Gap}(\Delta_0; \alpha, \beta)$  (see [26] for the definition of  $\text{Gap}(\Delta_0; \alpha, \beta)$ ).

In the theorem below, a generalized simple closed geodesic is a simple closed geodesic, a cone point, or a cusp.

**Theorem 4.4** ([26]). *Let  $X$  be a hyperbolic cone-surface with all cone angles in  $[0, \pi)$ , and  $\Delta_0$  a distinguished generalized boundary component. Then one has either*

$$(23) \quad \sum \text{Gap}(\Delta_0; \alpha, \beta) = \frac{\theta_0}{2},$$

when  $\Delta_0$  is a cone point of cone angle  $\theta_0$ ; or

$$(24) \quad \sum \text{Gap}(\Delta_0; \alpha, \beta) = \frac{l_0}{2},$$

when  $\Delta_0$  is a boundary geodesic of length  $l_0$ ; or

$$(25) \quad \sum \text{Gap}(\Delta_0; \alpha, \beta) = \frac{1}{2},$$

when  $\Delta_0$  is a cusp; where in each case the sum is over all pairs of generalized simple closed geodesics  $\alpha, \beta$  on  $M$  which bound with  $\Delta_0$  an embedded pair of pants.

**Remark 5.** For any non-peripheral simple closed geodesic  $\gamma$ , there is a generalized simple closed geodesic  $\gamma'$  with  $\gamma' \neq \gamma$ , such that  $\gamma, \gamma'$  and  $\Delta_0$  bounds an embedded pair of pants. Hence  $\gamma$  appears in the left side of each identity (see Fig. 3).

Combining Theorem 4.1 and Theorem 4.4, we get the following generalized marked length spectral rigidity.

**Theorem C** . Suppose that  $\Lambda \in (-\pi, \infty)^n$  and that  $S$  is a non-exceptional surface. Let  $X_1, X_2 \in \mathcal{T}_{g,n}(\Lambda)$ . If  $l_{X_1}([\alpha]) \geq l_{X_2}([\alpha])$  holds for every  $[\alpha] \in \mathcal{S}(S)$ , then  $X_1 = X_2$ .

*Proof.* First, we take a cone point  $P_0$  as the distinguished cone point  $\Delta_0$ . If there is no cone point, we pick a cusp or a geodesic boundary component instead. Since  $X_1, X_2 \in \mathcal{T}_{g,n}(\Lambda)$ , they share the same cone angles and boundary lengths, which lead to

$$(26) \quad \sum \text{Gap}_{X_1}(\Delta_0; \alpha, \beta) = \frac{\theta}{2} = \sum \text{Gap}_{X_2}(\Delta_0; \alpha, \beta),$$

where the sum is over all pairs of generalized simple closed geodesics  $\alpha, \beta$  on  $S$  which bound with  $P_0$  an embedded pair of pants.

Note that each *gap function* is a strictly decreasing function about the length of each involved non-peripheral simple closed geodesic. Combining this with the assumption that  $l_{X_1}(\alpha) \geq l_{X_2}(\alpha)$ , for any  $\alpha \in \mathcal{S}(S)$ , we have

$$(27) \quad \sum \text{Gap}_{X_1}(\Delta_0; \alpha, \beta) \leq \sum \text{Gap}_{X_2}(\Delta_0; \alpha, \beta),$$

with the equality holds if and only if  $l_{X_1}(\alpha) = l_{X_2}(\alpha)$  for any  $\alpha \in \mathcal{S}(S)$  (by Remark 5 every  $\alpha \in \mathcal{S}(S)$  is involved in the sum). Combining with (26) and (27), we have  $l_{X_1}(\alpha) = l_{X_2}(\alpha)$  for any  $\alpha \in \mathcal{S}(S)$ .

Now, the theorem follows directly from Theorem 4.3.  $\square$

As an application, we can define the Thurston's asymmetric metric on  $\mathcal{T}_{g,n}(\Lambda)$  for any  $\Lambda \in (-\pi, \infty)^n$ .

**Corollary 4.5** (the Thurston metric). *Suppose  $\Lambda \in (-\pi, \infty)^n$  and that  $S$  is a non-exceptional surface. For any two points  $X_1, X_2 \in \mathcal{T}_{g,n}(\Lambda)$ ,*

$$d_{Th}(X_1, X_2) \triangleq \log \sup_{[\alpha] \in \mathcal{S}(S)} \frac{l_{X_2}([\alpha])}{l_{X_1}([\alpha])}$$

*defines an asymmetric metric on  $\mathcal{T}_{g,n}(\Lambda)$ , which is called the Thurston metric.*

## 5. COMPARISONS OF GEOMETRY BETWEEN HYPERBOLIC CONE SURFACES

In this section, we build a connection between a general hyperbolic cone surface and a punctured hyperbolic surface by comparing the lengths of the geodesic representatives of isotopy classes of non-peripheral simple closed curves on both surfaces. Before that, we prove some lemmas first.

**Lemma 5.1.** *Assume  $c, c' > 0$  and let  $d(c, \rho_1, \rho_2), f(c, \rho_1, \rho_2)$  be two functions of  $\rho_1, \rho_2$  and  $c$  defined by*

$$\cosh d(c, \rho_1, \rho_2) = f(c, \rho_1, \rho_2) \triangleq \cosh \rho_1 \cosh \rho_2 \cosh c - \sinh \rho_1 \sinh \rho_2.$$

*If*

$$\frac{1}{C} \leq \frac{\cosh c - 1}{\cosh c' - 1} \leq C$$

*for some  $C > 1$ . Then*

(a)

$$|d(c', \rho_1, \rho_2) - d(c, \rho_1, \rho_2)| \leq \text{arccosh } C.$$

(b)

$$\frac{1}{C} \leq \frac{d(c, \rho_1, \rho_2)}{d(c', \rho_1, \rho_2)} \leq C.$$

*Proof.* Part (a) follows from Lemma 5.2(a) and 5.3(a), part (b) follows from Lemma 5.2(b) and 5.3(b).  $\square$

**Lemma 5.2.** *Let  $x, y > 0$ .*

(a) *If*

$$\frac{1}{C} \leq \frac{\cosh x}{\cosh y} \leq C,$$

*then*

$$|x - y| \leq \operatorname{arccosh} C.$$

(b) *If*

$$\frac{1}{C} \leq \frac{\sinh x}{\sinh y} \leq C,$$

*then*

$$\frac{1}{C} \leq \frac{x}{y} \leq C.$$

*Proof.* (a) Let  $F(y)$  be defined by

$$\cosh(y + F) = C \cosh y.$$

Hence

$$\frac{e^{y+F} + e^{-y-F}}{2} = C \cosh y,$$

then

$$F(y) = \ln(C \cosh y + \sqrt{C^2 \cosh^2 y - 1}) - y,$$

further,

$$F'(y) = \frac{C \sinh y}{\sqrt{C^2 \cosh^2 y - 1}} - 1 < 0.$$

As a result,

$$F(y) \leq F(0) = \operatorname{arccosh} C.$$

On the other hand,

$$\cosh x \leq C \cosh y = \cosh(y + F(y)).$$

It follows that

$$x - y \leq \operatorname{arccosh} C.$$

By symmetry, we have  $y - x \leq \operatorname{arccosh} C$ .

(b) Let  $K(y)$  be a function of  $y$  defined by

$$\sinh(yK) = C \sinh y.$$

Hence

$$\frac{e^{yK} - e^{-yK}}{2} = C \sinh y,$$

then

$$K(y) = \frac{\ln(C \sinh y + \sqrt{1 + C^2 \sinh^2 y})}{y},$$

further,

$$K'(y) = \frac{C \cosh y}{y \sqrt{1 + C^2 \sinh^2 y}} - \frac{\ln(C \sinh y + \sqrt{1 + C^2 \sinh^2 y})}{y^2}.$$

Setting  $K'(y_0) = 0$ , we have

$$\begin{aligned} K(y_0) &= \frac{\ln(C \sinh y_0 + \sqrt{1 + C^2 \sinh^2 y_0})}{y_0} \\ &= \frac{C \cosh y_0}{\sqrt{1 + C^2 \sinh^2 y_0}} \\ &< C. \end{aligned}$$

Note that

$$K(y) \rightarrow C, \text{ as } y \rightarrow 0 \quad \text{and} \quad K(y) \rightarrow 1, \text{ as } y \rightarrow \infty.$$

Therefore  $K(y) \leq C$  for any  $y \geq 0$ . Since  $\sinh x \leq C \sinh y = \sinh y K(y)$ ,  $x/y \leq C$ . By symmetry, we also have  $y/x \leq C$ .  $\square$

**Lemma 5.3.** *Assume  $0 < c \leq c'$  and let  $f(c, \rho_1, \rho_2)$  be the function defined in Lemma 5.1. then*

(a)

$$\frac{f(c, \rho_1, \rho_2)}{f(c', \rho_1, \rho_2)} \geq \frac{\cosh c - 1}{\cosh c' - 1}.$$

(b)

$$\frac{f(c, \rho_1, \rho_2)^2 - 1}{f(c', \rho_1, \rho_2)^2 - 1} \geq \left( \frac{\cosh c - 1}{\cosh c' - 1} \right)^2.$$

*Proof.* (a) Note that

$$\begin{aligned} \frac{f(c, \rho_1, \rho_2)}{f(c', \rho_1, \rho_2)} &= \frac{\cosh \rho_1 \cosh \rho_2 \cosh c - \sinh \rho_1 \sinh \rho_2}{\cosh \rho_1 \cosh \rho_2 \cosh c' - \sinh \rho_1 \sinh \rho_2} \\ &= 1 - \frac{\cosh c' - \cosh c}{\cosh c' - \tanh \rho_1 \tanh \rho_2} \\ &\geq 1 - \frac{\cosh c' - \cosh c}{\cosh c' - 1} \\ &= \frac{\cosh c - 1}{\cosh c' - 1}. \end{aligned}$$

(b) Observe that

$$\begin{aligned} \frac{f(c, \rho_1, \rho_2)^2 - 1}{f(c', \rho_1, \rho_2)^2 - 1} &= \frac{f(c, \rho_1, \rho_2) + 1}{f(c', \rho_1, \rho_2) + 1} \frac{f(c, \rho_1, \rho_2) - 1}{f(c', \rho_1, \rho_2) - 1} \\ &\geq \frac{f(c, \rho_1, \rho_2)}{f(c', \rho_1, \rho_2)} \frac{f(c, \rho_1, \rho_2) - 1}{f(c', \rho_1, \rho_2) - 1}, \end{aligned}$$

and

$$\begin{aligned} \frac{f(c, \rho_1, \rho_2) - 1}{f(c', \rho_1, \rho_2) - 1} &= 1 - \frac{\cosh c' - \cosh c}{\cosh c' - \frac{\sinh \rho_1 \sinh \rho_2 + 1}{\cosh \rho_1 \cosh \rho_2}} \\ &\geq 1 - \frac{\cosh c' - \cosh c}{\cosh c' - 1} \\ &\quad (\text{since } \cosh(\rho_1 - \rho_2) \geq 1) \\ &= \frac{\cosh c - 1}{\cosh c' - 1}. \end{aligned}$$

Then we have

$$\frac{f(c, \rho_1, \rho_2)^2 - 1}{f(c', \rho_1, \rho_2)^2 - 1} \geq \left( \frac{\cosh c - 1}{\cosh c' - 1} \right)^2.$$

□

Recall that a generalized boundary component  $\Delta$  is a cone point with cone angle  $\theta \in (0, \pi)$ , a cusp, or a geodesic boundary of positive length. The assignment  $\lambda(\Delta)$  at a generalized boundary component is:

$$(28) \quad \lambda(\Delta) = \begin{cases} -\theta, & \text{if } \Delta \text{ is a cone point of angle } \theta \in (0, \pi), \\ 0, & \text{if } \Delta \text{ is a cusp,} \\ l, & \text{if } \Delta \text{ is a geodesic boundary of length } l > 0. \end{cases}$$

Let  $V(\lambda_1, \lambda_2, \lambda_3)$  be a *generalised Y-piece* whose boundary assignments satisfy  $\Lambda = (\lambda_1, \lambda_2, \lambda_3) \in (0, \infty) \times (-\pi, \infty) \times (-\pi, \infty)$ , and  $\Delta_1, \Delta_2, \Delta_3$  be the three marked generalized boundary components. Let  $\zeta$  be an arbitrary simple arc with endpoints  $M, N$  on the geodesic boundary components (not a cone point nor a cusp). Assume that  $M$  is on  $\Delta_1$  and  $N$  is on  $\Delta^N \in \{\Delta_1, \Delta_2, \Delta_3\}$ . Denote by  $g_\zeta$  the geodesic representative in the corresponding homotopy class of  $\zeta$  relative to the endpoints, i.e. the endpoints  $M, N$  stay fixed during the homotopy, and by  $h_\zeta$  the geodesic representative in the homotopy class of  $\zeta$  relative to the geodesic boundaries of  $V(\lambda_1, \lambda_2, \lambda_3)$ , i.e. the endpoints  $M, N$  stay at the geodesic boundaries during the homotopy process. Let  $l(\lambda_1, \lambda_2, \lambda_3, [\zeta])$  be the length of  $g_\zeta$ . Let  $h_\Lambda$  (resp.  $h_0$ ) be the length of  $h_\zeta$  on  $V(\lambda_1, \lambda_2, \lambda_3)$  (resp.  $V(\lambda_1, \lambda_2, 0)$  or  $V(\lambda_1, 0, 0)$ , up to the situation in consideration).

Let the geodesic boundary  $\Delta_1, \Delta^N$  be oriented such that  $V(\lambda_1, \lambda_2, \lambda_3)$  sits on the right of  $\Delta_1$  and on the left of  $\Delta^N$ . Since  $g_\zeta$  and  $h_\zeta$  are homotopic relative to the boundaries of  $V(\lambda_1, \lambda_2, \lambda_3)$ , there is a homotopy  $H$  between  $g_\zeta$  and  $h_\zeta$  on  $V(\lambda_1, \lambda_2, \lambda_3)$ . Denote by  $\rho_M$  and  $\rho_N$  be the signed displacements of  $h_\zeta \cap \Delta_1$  and  $h_\zeta \cap \Delta^N$  during the homotopy process.

**Remark 6.** The assumption  $\Lambda = (\lambda_1, \lambda_2, \lambda_3) \in (0, \infty) \times (-\pi, \infty) \times (-\pi, \infty)$  guarantees that  $\Delta_1$  is a geodesic boundary. The reason for this assumption is that the surface we are interested is not a sphere with three holes, which means that each pair of pants on the surface has at least one boundary component which is a non-peripheral simple closed curve.

**Lemma 5.4.** *With notations above, we have*

$$\cosh l(\lambda_1, \lambda_2, \lambda_3, [\zeta]) = \cosh \rho_M \cosh \rho_N \cosh h_\Lambda - \sinh \rho_M \sinh \rho_N.$$

*Proof.* We consider the case that  $\Lambda = (\lambda_1, \lambda_2, \lambda_3) \in (0, \infty) \times (0, \infty) \times (-\pi, \infty)$ , the proofs for the remaining cases are similar.

Recall that  $V(\lambda_1, \lambda_2, \lambda_3)$  consists of two isometric pentagons pictured in Figure 6(b). Extend the geodesic  $g_\zeta$  on  $V(\lambda_1, \lambda_2, \lambda_3)$  to the hyperbolic plane as illustrated in Fig.6(c). Then the lemma follows from equation (8) for the quadrilateral.  $\square$

The lemma below describes the comparisons of the geometry between a pair of general pants and a pair of pants which do not contain any cone point.

**Lemma 5.5.** *With the notations above, we have the following estimates:*

- (a) *If  $\Delta_1, \Delta_2$  are geodesic boundaries,  $\Delta_3$  is a generalized geodesic boundary, then there exist  $C_1, D_1 > 0$ , which depend only on  $\lambda_3$ , such that*

$$|l(\lambda_1, \lambda_2, \lambda_3, [\zeta]) - l(\lambda_1, \lambda_2, 0, [\zeta])| \leq D_1,$$

and

$$\frac{1}{C_1} \leq \frac{l(\lambda_1, \lambda_2, \lambda_3, [\zeta])}{l(\lambda_1, \lambda_2, 0, [\zeta])} \leq C_1.$$

Moreover,  $D_1 \rightarrow 0, C_1 \rightarrow 1$  as  $\lambda_3 \rightarrow 0$ .

- (b) *If  $\Delta_1$  is a geodesic boundary,  $\Delta_2$  and  $\Delta_3$  are generalized geodesic boundaries with the same type, then there exist  $C_2, D_2 > 0$ , which depend only on  $\lambda_2, \lambda_3$ , such that*

$$|l(\lambda_1, \lambda_2, \lambda_3, [\zeta]) - l(\lambda_1, 0, 0, [\zeta])| \leq D_2,$$

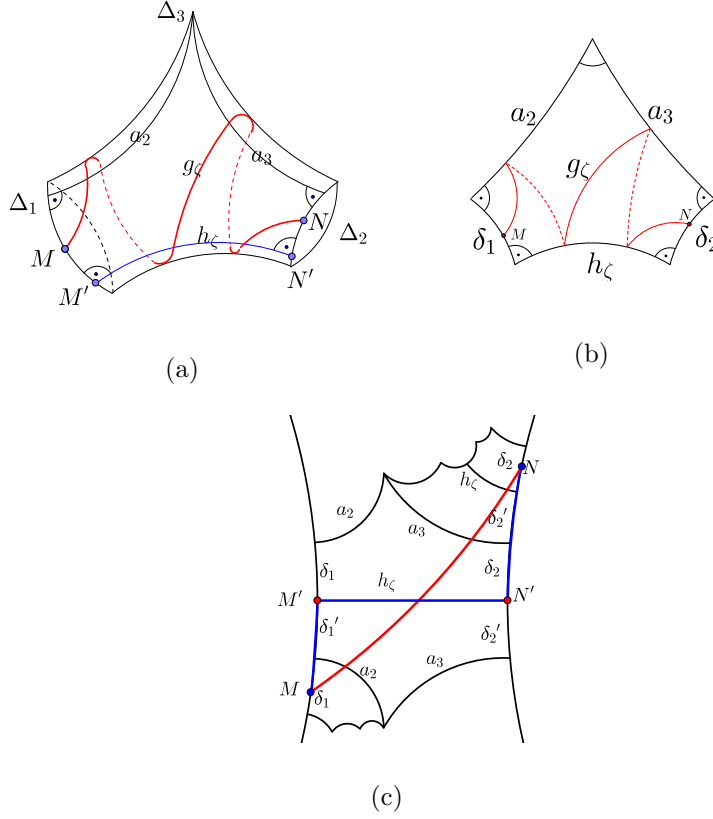


FIGURE 6. Figure (c) pictures the extension of  $g_\zeta$  to the hyperbolic plane. The V-piece in Figure (a) consists of two isometric pentagons in Figure (b). If we record the edges of these two pentagons meet by  $g_\zeta$  during the extension, the geodesic  $g_\zeta$  in Figure (a) corresponds to a sequence  $(\delta_1, a_2, h_\zeta, a_3, h_\zeta, \delta_2)$ .

and

$$\frac{1}{C_2} \leq \frac{l(\lambda_1, \lambda_2, \lambda_3, [\zeta])}{l(\lambda_1, 0, 0, [\zeta])} \leq C_2.$$

Moreover,  $D_2 \rightarrow 0, C_2 \rightarrow 1$  as  $\lambda_2, \lambda_3 \rightarrow 0$ .

(c) If  $\Delta_1$  is a geodesic boundary,  $\Delta_2$  and  $\Delta_3$  are generalized geodesic boundaries with different types, then there exist  $C_3, D_3 > 0$ , which depend only on  $\lambda_2, \lambda_3$ , such that

$$|l(\lambda_1, \lambda_2, \lambda_3, [\zeta]) - l(\lambda_1, 0, 0, [\zeta])| \leq D_3,$$



and

$$\frac{1}{C_3} \leq \frac{l(\lambda_1, \lambda_2, \lambda_3, [\zeta])}{l(\lambda_1, 0, 0, [\zeta])} \leq C_3.$$

Moreover,  $D_3 \rightarrow 0, C_3 \rightarrow 1$ , as  $\lambda_2 \rightarrow 0, \lambda_3 \rightarrow 0$ .

- Remark 7.** (i) Here the point is that the estimates have nothing to do with the homotopy classes of  $\zeta$ .  
(ii) The estimates about the difference of corresponding lengths works for the situation where the lengths tend to infinity, while the estimates about the ratio of corresponding lengths works for the situation where the lengths tend to zero. Both of them will be used to prove the quasi-isometry of  $\mathcal{T}_{g,n}(\Lambda)$  and  $\mathcal{T}_{g,n}(0)$  and to solve various boundary problems.

*Proof.* For simplicity, we denote by  $h_\Lambda$  (resp.  $h_0$ ) the length of  $h_\zeta$  on  $V(\lambda_1, \lambda_2, \lambda_3)$  (resp.  $V(\lambda_1, \lambda_2, 0)$  or  $V(\lambda_1, 0, 0)$ , up to the situation in consideration).

It follows from Lemma 5.4 that

$$\begin{cases} \cosh l(\lambda_1, \lambda_2, \lambda_3, [\zeta]) &= \cosh \rho_M \cosh \rho_N \cosh h_\Lambda - \sinh \rho_M \sinh \rho_N \\ \cosh l(\lambda_1, \lambda_2, 0, [\zeta]) &= \cosh \rho_M \cosh \rho_N \cosh h_0 - \sinh \rho_M \sinh \rho_N \end{cases}$$

for statement (a), and

$$\begin{cases} \cosh l(\lambda_1, \lambda_2, \lambda_3, [\zeta]) &= \cosh \rho_M \cosh \rho_N \cosh h_\Lambda - \sinh \rho_M \sinh \rho_N \\ \cosh l(\lambda_1, 0, 0, [\zeta]) &= \cosh \rho_M \cosh \rho_N \cosh h_0 - \sinh \rho_M \sinh \rho_N \end{cases}$$

for statement (b),(c).

By Lemma 5.1, it suffices to prove that

$$\frac{1}{K} \leq \frac{\cosh h_\Lambda - 1}{\cosh h_0 - 1} \leq K$$

for some  $K$  that approaches 1 as we take the limit for each statement in the lemma.

We distinguish six cases. The statement (a) is obtained from the cases A-D, and the statement (b) is obtained from the cases E and F. By statement (a) and statement (b), we have statement (c).

**Case A.**  $\Delta_1, \Delta_2$  are geodesic boundaries,  $\Delta_3$  is a cone point or a cusp and  $N \in \Delta_2$  (see Fig. 6(a)).

Using equation (6) for the pentagon pictured in Fig. 6(b), we have

$$\cosh h_\Lambda = \frac{\cosh \frac{\lambda_1}{2} \cosh \frac{\lambda_2}{2} + \cos \frac{|\lambda_3|}{2}}{\sinh \frac{\lambda_1}{2} \sinh \frac{\lambda_2}{2}},$$

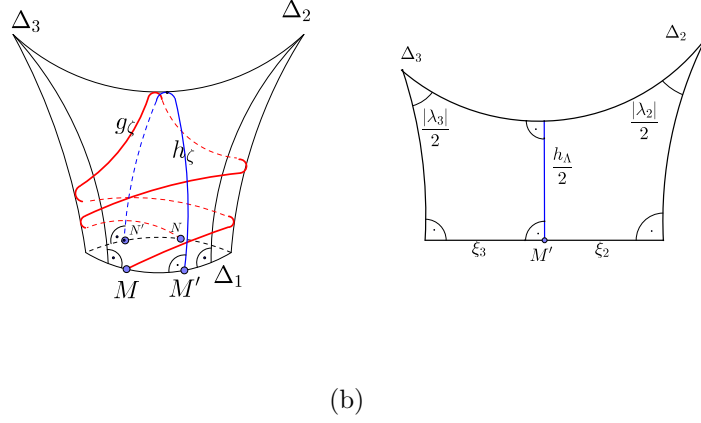
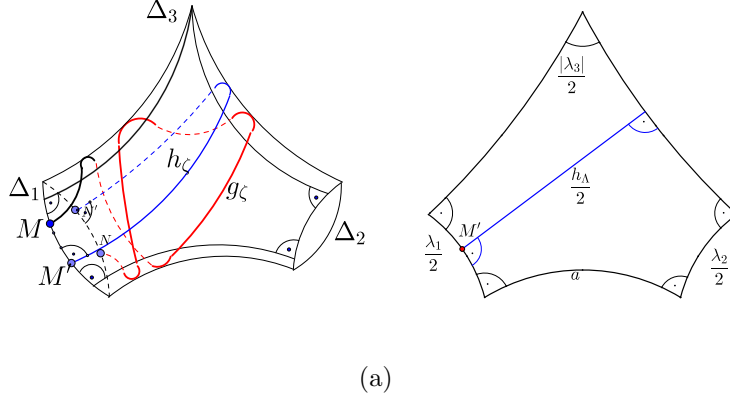


FIGURE 7. The arc  $\zeta$  in the figures on the left are arbitrary geodesic arcs with endpoint on given boundary components, and  $h_\zeta$  is the geodesic homotopic to  $\zeta$  relative to the boundary components of  $V$ . Each figure on the right side is one of the two isometric polygons consisting the corresponding pair of pants on its left.

where  $\lambda_1$  and  $\lambda_2$  are the lengths of the geodesic boundaries  $\Delta_1$  and  $\Delta_2$  respectively,  $|\lambda_3|$  is the cone angle of  $\Delta_3$ . Hence

$$\frac{\cosh h_\Lambda - 1}{\cosh h_0 - 1} = \frac{\cosh \frac{\lambda_1 - \lambda_2}{2} + \cos \frac{|\lambda_3|}{2}}{\cosh \frac{\lambda_1 - \lambda_2}{2} + 1} \in \left[ \frac{1 + \cos \frac{|\lambda_3|}{2}}{2}, 1 \right].$$

**Case B.**  $\Delta_1, \Delta_2$  are geodesic boundaries,  $\Delta_3$  is a geodesic boundary and  $N \in \Delta_2$ .

Replace  $\cos \frac{|\lambda_3|}{2}$  in case A by  $\cosh \frac{|\lambda_3|}{2}$ , we have

$$\frac{\cosh h_\Lambda - 1}{\cosh h_0 - 1} = \frac{\cosh \frac{\lambda_1 - \lambda_2}{2} + \cosh \frac{|\lambda_3|}{2}}{\cosh \frac{\lambda_1 - \lambda_2}{2} + 1} \in [1, \frac{1 + \cosh \frac{|\lambda_3|}{2}}{2}].$$

**Case C.**  $\Delta_1, \Delta_2$  are geodesic boundaries,  $\Delta_3$  is a cone point or a cusp and  $N \in \Delta_1$  (see Fig. 7(a)).

Using equation (5) for the right angled pentagon pictured in Fig. 7(a), we have

$$\cosh \frac{h_\Lambda}{2} = \sinh \frac{\lambda_2}{2} \sinh a,$$

where  $a$  is the length of the geodesic perpendicular to  $\Delta_1$  and  $\Delta_2$ .

$$\begin{aligned} \cosh h_\Lambda - 1 &= 2 \cosh^2 \frac{h_\Lambda}{2} - 2 \\ &= 2 \frac{\cosh^2 \frac{\lambda_2}{2} + 2 \cosh \frac{\lambda_1}{2} \cosh \frac{\lambda_2}{2} \cos \frac{|\lambda_3|}{2} + \cos^2 \frac{|\lambda_3|}{2}}{\sinh^2 \frac{\lambda_1}{2}}, \end{aligned}$$

and

$$\begin{aligned} \frac{\cosh h_\Lambda - 1}{\cosh h_0 - 1} &= \frac{\cosh^2 \frac{\lambda_2}{2} + 2 \cosh \frac{\lambda_1}{2} \cosh \frac{\lambda_2}{2} \cos \frac{|\lambda_3|}{2} + \cos^2 \frac{|\lambda_3|}{2}}{\cosh^2 \frac{\lambda_2}{2} + 2 \cosh \frac{\lambda_1}{2} \cosh \frac{\lambda_2}{2} + 1} \\ &\in [\cos^2 \frac{|\lambda_3|}{2}, 1]. \end{aligned}$$

**Case D.**  $\Delta_1, \Delta_2$  are geodesic boundaries,  $\Delta_3$  is a geodesic boundary and  $N \in \Delta_1$ .

Replace  $\cos \frac{|\lambda_3|}{2}$  in case C by  $\cosh \frac{|\lambda_3|}{2}$ , we have

$$\begin{aligned} \frac{\cosh h_\Lambda - 1}{\cosh h_0 - 1} &= \frac{\cosh^2 \frac{\lambda_2}{2} + 2 \cosh \frac{\lambda_1}{2} \cosh \frac{\lambda_2}{2} \cosh \frac{|\lambda_3|}{2} + \cosh^2 \frac{|\lambda_3|}{2}}{\cosh^2 \frac{\lambda_2}{2} + 2 \cosh \frac{\lambda_1}{2} \cosh \frac{\lambda_2}{2} + 1} \\ &\in [1, \cosh^2 \frac{|\lambda_3|}{2}]. \end{aligned}$$

**Case E.**  $\Delta_1$  is a geodesic boundary,  $\Delta_2, \Delta_3$  are cone points or cusps, and  $N \in \Delta_1$  (see Fig. 7(b)).

We choose a quadrilateral with two adjacent vertices at  $\Delta_1$  and  $\Delta_2$ , and two right angles at the endpoints of the opposite side  $\xi$  contained

in  $\Delta_1$  (see the right figure of Fig. 7(b)). Let  $\xi_2, \xi_3$  be the subsides of  $\xi$  divided by  $h_\zeta$ . From (4) in Lemma 2.3, we have

$$\cos \frac{|\lambda_2|}{2} = \sinh \frac{h_\Lambda}{2} \sinh |\xi_2|, \quad \cos \frac{|\lambda_3|}{2} = \sinh \frac{h_\Lambda}{2} \sinh |\xi_3|.$$

Set  $l_\Lambda \triangleq (\sinh \frac{h_\Lambda}{2})^{-1}$ , then

$$\sinh |\xi_2| = l_\Lambda \cos \frac{|\lambda_2|}{2}, \quad \sinh |\xi_3| = l_\Lambda \cos \frac{|\lambda_3|}{2}.$$

Note that  $|\xi_2| + |\xi_3| = \lambda_1/2$ , hence

$$\sinh \frac{\lambda_1}{2} = l_\Lambda \cos \frac{|\lambda_2|}{2} [1 + l_\Lambda^2 \cos^2 \frac{|\lambda_3|}{2}]^{1/2} + l_\Lambda \cos \frac{|\lambda_3|}{2} [1 + l_\Lambda^2 \cos^2 \frac{|\lambda_2|}{2}]^{1/2}.$$

Set

$$K \triangleq \max\{\cos \frac{|\lambda_2|}{2}, \cos \frac{|\lambda_3|}{2}\}, \quad k \triangleq \min\{\cos \frac{|\lambda_2|}{2}, \cos \frac{|\lambda_3|}{2}\}.$$

Then we get

$$2kl_\Lambda[1 + k^2l_\Lambda^2]^{1/2} \leq \sinh \frac{\lambda_1}{2} \leq 2Kl_\Lambda[1 + K^2l_\Lambda^2]^{1/2},$$

which means that

$$\frac{\sinh \frac{\lambda_1}{4}}{K} \leq l_\Lambda \leq \frac{\sinh \frac{\lambda_1}{4}}{k}.$$

In particular,

$$l_0 = \sinh \frac{\lambda_1}{4}.$$

Therefore

$$\begin{aligned} \frac{\cosh h_\Lambda - 1}{\cosh h_0 - 1} &= \frac{\sinh^2 \frac{h_\Lambda}{2}}{\sinh^2 \frac{h_0}{2}} = \frac{l_0^2}{l_\Lambda^2} \\ &\in [k^2, K^2]. \end{aligned}$$

**Case F.**  $\Delta_1$  is a geodesic boundary,  $\Delta_2, \Delta_3$  are geodesic boundaries, and  $N \in \Delta_1$ .

Replace  $\cos \frac{|\lambda_2|}{2}$  and  $\cos \frac{|\lambda_3|}{2}$  in case E by  $\cosh \frac{|\lambda_2|}{2}$  and  $\cosh \frac{|\lambda_3|}{2}$ , we get

$$\begin{aligned} \frac{\cosh h_\Lambda - 1}{\cosh h_0 - 1} &= \frac{\sinh^2 \frac{h_\Lambda}{2}}{\sinh^2 \frac{h_0}{2}} = \frac{l_0^2}{l_\Lambda^2} \\ &\in [k^2, K^2], \end{aligned}$$

where

$$K \triangleq \max\{\cosh \frac{|\lambda_2|}{2}, \cosh \frac{|\lambda_3|}{2}\}, \quad k \triangleq \min\{\cosh \frac{|\lambda_2|}{2}, \cosh \frac{|\lambda_3|}{2}\}.$$

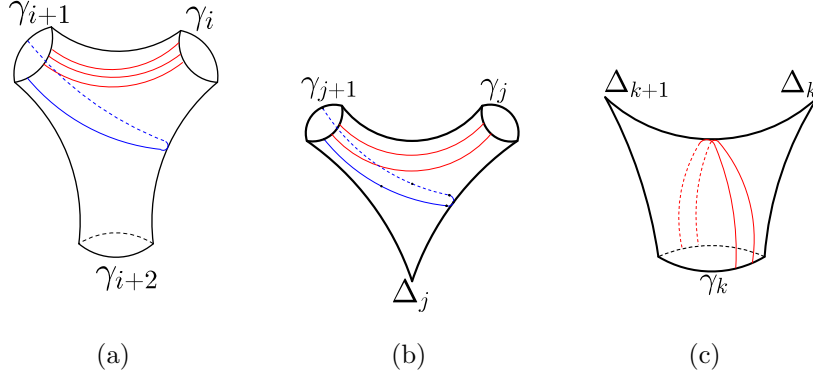


FIGURE 8.

□

**Remark 8.** The estimates in Lemma 5.5 are optimal since all the equalities in the inequations can hold simultaneously.

Next, we compare the lengths of non-peripheral simple closed curves between hyperbolic cone surfaces based on Lemma 5.5.

**Theorem B** (Length comparison inequalities). Let  $S$  be a non-exceptional surface. There exist constants  $C, D$  depending on  $\Lambda$  such that for any  $X \in \mathcal{T}_{g,n}(\Lambda)$ ,  $X' = F_{\Gamma, \mathcal{B}, \Lambda}(X)$  and any isotopy class of non-peripheral simple closed curve  $[\alpha]$ ,

$$\begin{cases} |l_X([\alpha]) - l_{X'}([\alpha])| \leq D \sum_{j=1}^{3g-3+n} i([\alpha], [\gamma_j]), \\ \frac{1}{C} \leq \frac{l_X([\alpha])}{l_{X'}([\alpha])} \leq C, \end{cases}$$

where  $i(\cdot, \cdot)$  is the geometric intersection number, and  $l_X([\alpha])$  is the length of the geodesic representative in  $[\alpha]$ . Moreover,  $D \rightarrow 0, C \rightarrow 1$  as  $\Lambda \rightarrow 0$ .

*Proof.* The pants decomposition  $\Gamma$  divides the surface  $S$  into  $2g - 2 + n$  pairs of topological pants  $\mathcal{R} = \{R_1, \dots, R_{2g-2+n}\}$ . Let  $R_i^X$  be the restriction of  $X$  on  $R_i$ ,  $\alpha^X$  the geodesic representative of  $[\alpha]$  on  $X$ , and  $\alpha_i^X$  the union of restrictions of  $\alpha^X$  on  $R_i^X$ ,  $i = 1, \dots, 2g - 2 + n$  (see Fig. 8). The sum of the numbers of arcs in  $\alpha_i^X$  on all pairs of pants is  $\sum_{j=1}^{3g-3+n} i([\alpha], [\gamma_j])$ . For any different  $X, X' \in \mathcal{T}_{g,n}(\Lambda)$ , the geodesic representative  $\alpha^X$  of  $\alpha$  on  $X$  is usually not a geodesic representative of  $\alpha$  on  $X'$ . We modify  $\alpha^X$  as follows: for any  $i = 1, \dots, 2g - 2 + n$ , replace each arc in  $\alpha_i^X$  by its geodesic representative with respect to  $X'$  in its homotopy class relative to the endpoints. Denote by  $\alpha^{XX'}$  the resulting

simple closed curve and by  $\alpha_i^{XX'}$  the restriction of  $\alpha^{XX'}$  on  $R_i^{X'}$ . Recall that  $l_X([\alpha])$  represents the length of the geodesic representative in  $[\alpha]$ , we denote by  $l_X(k)$  the length of any arc  $k$  on  $X$ .

(a) From Lemma 5.5, we have

$$\begin{aligned} l_{X'}([\alpha]) - l_X([\alpha]) &\leq l_{X'}(\alpha^{XX'}) - l_X([\alpha]) \\ &= \sum_{i=1}^{2g-2+n} [l_{X'}(\alpha_i^{XX'}) - l_X(\alpha_i^X)] \\ &\leq D \sum_{j=1}^{3g-3+n} i([\alpha], [\gamma_j]), \end{aligned}$$

where  $D = \max\{D_1, D_2, D_3\}$  and  $D_1, D_2, D_3$  are the constants in Lemma 5.5.

By interchanging the role of  $X$  and  $X'$ , we get

$$l_X([\alpha]) - l_{X'}([\alpha]) \leq D \sum_{j=1}^{3g-3+n} i([\alpha], [\gamma_j]).$$

Moreover, we have  $D \rightarrow 0$ , as  $\Lambda \rightarrow 0$ .

(b) Again from Lemma 5.5, we have

$$\begin{aligned} l_{X'}([\alpha]) &\leq l_{X'}(\alpha^{XX'}) \\ &= \sum_{i=1}^{2g-2+n} l_{R_i^{X'}}(\alpha_i^{XX'}) \\ &\leq \sum_{i=1}^{2g-2+n} C l_{R_i^X}(\alpha_i^X) \\ &= C l_X(\alpha^X) \\ &= C l_X([\alpha]), \end{aligned}$$

and

$$\begin{aligned} l_{X'}([\alpha]) &= l_{X'}(\alpha^{X'}) \\ &= \sum_{i=1}^{2g-2+n} l_{R_i^{X'}}(\alpha_i^{X'}) \\ &\geq \sum_{i=1}^{2g-2+n} C^{-1} l_{R_i^X}(\alpha_i^{X'X}) \\ &= C^{-1} l_X(\alpha^{X'X}) \\ &\geq C^{-1} l_X([\alpha]), \end{aligned}$$

where  $C = \max\{C_1, C_2, C_3\}$  and  $C_1, C_2, C_3$  are the constants in Lemma 5.5.

Moreover  $C \rightarrow 1$ , as  $\Lambda \rightarrow 0$ .

□

## 6. ALMOST-ISOMETRY BETWEEN $\mathcal{T}_{g,n}(\Lambda)$ AND $\mathcal{T}_{g,n}(0)$

In this section, we prove Theorem D.

**Theorem D** (Almost-isometry). Suppose  $\Lambda \in (-\pi, \infty)^n$  and that  $S$  is a non-exceptional surface. Then  $F_{\Gamma, \mathcal{B}, \Lambda} : \mathcal{T}_{g,n}(\Lambda) \rightarrow \mathcal{T}_{g,n}(0)$  is an

almost-isometry, i.e. there is a constant  $C$  depending only on  $\Lambda$  such that

$d_{Th}(X'_1, X'_2) - C \leq d_{Th}(X_1, X_2) \leq d_{Th}(X'_1, X'_2) + C$ , for any  $X_1, X_2 \in \mathcal{T}_{g,n}$ , where  $X'_1 = F_{\Gamma, \Lambda}(X_1)$  and  $X'_2 = F_{\Gamma, \Lambda}(X_2)$ . Moreover,  $C \rightarrow 0$ , as  $\Lambda \rightarrow 0$ .

*Proof.* It follows from Theorem B that

$$\begin{aligned} d_{Th}(X_1, X_2) &= \sup_{[\alpha] \in \mathcal{S}(S)} \log \frac{l_{X_2}([\alpha])}{l_{X_1}([\alpha])} \\ &\leq \sup_{[\alpha] \in \mathcal{S}(S)} \log \frac{Cl_{X'_2}([\alpha])}{C^{-1}l_{X'_1}([\alpha])} \\ &= d_{Th}(X'_1, X'_2) + 2 \log C, \end{aligned}$$

and

$$\begin{aligned} d_{Th}(X_1, X_2) &= \sup_{[\alpha] \in \mathcal{S}(S)} \log \frac{l_{X_2}([\alpha])}{l_{X_1}([\alpha])} \\ &\geq \sup_{[\alpha] \in \mathcal{S}(S)} \log \frac{C^{-1}l_{X'_2}([\alpha])}{Cl_{X'_1}([\alpha])} \\ &= d_{Th}(X'_1, X'_2) - 2 \log C. \end{aligned}$$

□

It follows from Remark 3 that the map  $F_{\Gamma, \mathcal{B}, \Lambda}$  is independent of  $\mathcal{B}$ . Instead, it depends on the choice of  $\Gamma$ , but the images of  $X \in \mathcal{T}_{g,n}(\Lambda)$  under different pants-compositions  $\Gamma, \Gamma'$  stay within a bounded distance, i.e.

$$d_{Th}(F_{\Gamma, \mathcal{B}, \Lambda}(X), F_{\Gamma', \mathcal{B}', \Lambda}(X)) \leq 2C,$$

where  $\mathcal{B}'$  is a collection of seams with respect to  $\Gamma'$ .

Indeed, it follows from Theorem B that for any non-peripheral simple closed curve  $\alpha$ , we have

$$\frac{1}{C} \leq \frac{l_X([\alpha])}{l_{F_{\Gamma, \mathcal{B}, \Lambda}(X)}([\alpha])} \leq C$$

and

$$\frac{1}{C} \leq \frac{l_X([\alpha])}{l_{F_{\Gamma', \mathcal{B}', \Lambda}(X)}([\alpha])} \leq C,$$

where  $C$  is the constant from Theorem B. Hence,

$$\frac{1}{C} \leq \frac{l_{F_{\Gamma', \mathcal{B}', \Lambda}(X)}([\alpha])}{l_{F_{\Gamma, \mathcal{B}, \Lambda}(X)}([\alpha])} \leq C,$$

which induces

$$d_{Th}(F_{\Gamma, \mathcal{B}, \Lambda}(X), F_{\Gamma', \mathcal{B}', \Lambda}(X)) \leq 2C.$$

Besides, the composition map  $F_{\Gamma, \mathcal{B}, \Lambda} \circ F_{\Gamma', \mathcal{B}', \Lambda}^{-1} : \mathcal{T}_{g,n}(0) \rightarrow \mathcal{T}_{g,n}(0)$  defines an almost-isometry from  $\mathcal{T}_{g,n}(0)$  to itself.

## 7. BASIC PROPERTIES OF $\mathcal{T}_{g,n}(\Lambda)$

In this section, we suppose that  $\Lambda \in (-\pi, \infty)^n$  and that the surfaces are admissible.

**7.1. Topology on  $\mathcal{T}_{g,n}(\Lambda)$ .** From the discussions in sections §2-§4, we know that the Teichmüller space  $\mathcal{T}_{g,n}(\Lambda)$  can be equipped with the following topologies:

- the topology induced from the functional space  $\mathbb{R}_+^S$  over the space of isotopy classes of non-peripheral simple closed curves, where  $\mathbb{R}_+^S$  is endowed with the weak topology;
- the topology induced from the Fenchel-Nielsen coordinates;
- the topology induced from the Thurston metric.

It follows from Theorem 4.1 that the first topology and the second topology are equivalent. The equivalence between the first topology and the third topology can be obtained from Theorem E.

**7.2. Metric properties of  $(\mathcal{T}_{g,n}(\Lambda), d_{Th})$ .** Let  $\bar{d}_{Th}(X_1, X_2) \triangleq d_{Th}(X_2, X_1)$  and  $d_{sym}(X_1, X_2) \triangleq d_{Th}(X_1, X_2) + d_{Th}(X_2, X_1)$ . We have the following basic properties.

**Proposition 7.1.** *Suppose  $\Lambda \in (-\pi, \infty)^n$  and that  $S$  is a non-exceptional surface. Let  $(\mathcal{T}_{g,n}(\Lambda), d_{Th})$  be endowed with the topology induced by the metric  $d_{Th}$ . Then*

- (a)  $d_{Th}$  is a proper metric on  $\mathcal{T}_{g,n}(\Lambda)$ .
- (b)  $(\mathcal{T}_{g,n}(\Lambda), d_{Th})$  is a geodesic metric space.
- (c) The topologies on  $\mathcal{T}_{g,n}(\Lambda)$  defined by  $d_{Th}$ ,  $\bar{d}_{Th}$  and  $d_{sym}$  are the same,  
i.e.  $d_{Th}(X, X_n) \rightarrow 0 \iff d_{Th}(X_n, X) \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof.* (a) Since  $\mathcal{T}_{g,n}(\Lambda)$  is homeomorphic to  $(-\pi, \infty)^n \times \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n}$ , a subset of  $\mathcal{T}_{g,n}(\Lambda)$  is compact if and only if it is closed and bounded. It follows that  $d_{Th}$  is a proper metric.

(b) Suppose  $X_1, X_2 \in \mathcal{T}_{g,n}(\Lambda)$ , it follows from (a) that the ball  $\{X \in \mathcal{T}_{g,n}(\Lambda) : d(X_1, X) \leq d(X_1, X_2)\}$  is a compact subset of  $\mathcal{T}_{g,n}(\Lambda)$ . Now the existence of a geodesic connecting  $X_1$  and  $X_2$  in  $\{X \in \mathcal{T}_{g,n}(\Lambda) : d(X_1, X) \leq d(X_1, X_2)\}$  is a standard argument by applying the Ascoli-Arzelà Theorem (see §1.6 in [1] for instance).



(c) The proof is similar to that in [12] and [19].

□

**7.3. The Thurston's boundary of  $\mathcal{T}_{g,n}(\Lambda)$ .** A *measured geodesic lamination* is a geodesic lamination equipped with a transverse invariant measure (see [21] for the precise definition of measured laminations). The simplest example of a measured geodesic lamination is a weighted simple closed curve. A measured geodesic lamination  $(L, \lambda)$  can be viewed as a functions on the set of isotopy classes of non-peripheral simple closed curves via:

$$\begin{aligned} i_{(L, \lambda)} : \mathcal{S}(S) &\longrightarrow [0, \infty) \\ [\alpha] &\longmapsto \inf_{\bar{\alpha} \in [\alpha]} \lambda(\bar{\alpha}). \end{aligned}$$

The value  $\inf_{\bar{\alpha} \in [\alpha]} \lambda(\bar{\alpha})$  is called the *intersection number* of  $(L, \lambda)$  and  $[\alpha]$ . From now on, for simplicity, we just write  $\lambda$  instead of  $(L, \lambda)$  and denote the intersection number between  $\lambda$  and  $[\alpha]$  by  $i(\lambda, [\alpha])$ . Two measured geodesic laminations  $\lambda'$  and  $\lambda$  are called *equivalent* if  $i(\lambda, [\alpha]) = i(\lambda', [\alpha])$  for all  $[\alpha] \in \mathcal{S}(S)$ . Denote by  $\mathcal{ML}(S)$  the space of equivalence classes of measured geodesic laminations. The definition of intersection number induces an embedding of  $\mathcal{ML}(S)$  into  $\mathbb{R}_+^{\mathcal{S}}$ , the functional space of  $\mathcal{S}(S)$ . Let  $\mathcal{PML}(S) \triangleq \mathcal{ML}(S)/\mathbb{R}_+$  be the projection of  $\mathcal{ML}(S)$ .

**Theorem 7.2** ([7] The Thurston's boundary of  $\mathcal{T}_{g,n}(0)$ ). *Let  $\Psi_0$  and  $\Pi$  be the maps defined as following:*

$$\begin{aligned} \Psi_0 : \mathcal{T}_{g,n}(0) &\longrightarrow \mathbb{R}_+^{\mathcal{S}} \\ X &\longmapsto (l_X([\alpha]))_{\alpha \in \mathcal{S}(S)}, \end{aligned}$$

and

$$\begin{aligned} \Pi : \mathbb{R}_+^{\mathcal{S}} &\longrightarrow P\mathbb{R}_+^{\mathcal{S}} \\ (s_\alpha)_{\alpha \in \mathcal{S}(S)} &\longmapsto [(s_\alpha)_{\alpha \in \mathcal{S}(S)}]. \end{aligned}$$

Then

- (a) both  $\Psi_0$  and  $\Pi \circ \Psi_0$  are embeddings, where  $\mathcal{T}_{g,n}(0)$  is equipped with the topology induced by  $d_{Th}$  and  $\mathbb{R}_+^{\mathcal{S}}$  is equipped with the weak topology, i.e. the pointwise convergence topology;
- (b) the boundary of  $\mathcal{T}_{g,n}(0)$  in  $P\mathbb{R}_+^{\mathcal{S}}$  is the projection of the space of measured lamination, i.e.  $\partial \overline{\mathcal{T}_{g,n}(0)}^{Th} = \mathcal{PML}(S)$ .

**Theorem E** (The Thurston's boundary). Suppose  $\Lambda = (\lambda_1, \dots, \lambda_n) \in (-\pi, \infty)^n$  and that  $S$  is a non-exceptional surface. Let  $\Psi_\Lambda$  and  $\Pi$  be the maps defined below:

$$\begin{aligned} \Psi_\Lambda : \mathcal{T}_{g,n}(\Lambda) &\longrightarrow \mathbb{R}_+^s \\ X &\longmapsto (l_X([\alpha]))_{\alpha \in \mathcal{S}(S)}, \end{aligned}$$

and

$$\begin{aligned} \Pi : \mathbb{R}_+^s &\longrightarrow P\mathbb{R}_+^s \\ (s_\alpha)_{\alpha \in \mathcal{S}(S)} &\longmapsto [(s_\alpha)_{\alpha \in \mathcal{S}(S)}]. \end{aligned}$$

Then the followings hold.

- (a) Both  $\Psi_\Lambda$  and  $\Pi \circ \Psi_\Lambda$  are embeddings, where  $\mathcal{T}_{g,n}(\Lambda)$  is equipped with the topology induced by  $d_{Th}$  and  $\mathbb{R}_+^s$  is equipped with the weak topology, i.e. the pointwise convergence topology;
- (b) Denote by  $\overline{\partial \mathcal{T}_{g,n}(\Lambda)}^{Th}$  the closure of  $\mathcal{T}_{g,n}(\Lambda)$  in  $P\mathbb{R}_+^s$ .  $\mathcal{T}_{g,n}(\Lambda) \ni X_n \longrightarrow \xi \in \overline{\partial \mathcal{T}_{g,n}(\Lambda)}^{Th} \iff \mathcal{T}_{g,n}(0) \ni F_{\Gamma, \Lambda}(X_n) \longrightarrow \xi \in \mathcal{PML}(S)$ , as a result, the boundary of  $\mathcal{T}_{g,n}(\Lambda)$  in  $P\mathbb{R}_+^s$  is the projection of the space of measured lamination, i.e.  $\overline{\partial \mathcal{T}_{g,n}(\Lambda)}^{Th} = \mathcal{PML}(S)$ .

*Proof.* (a) Suppose that there is at least one cone point on the surface which we denote by  $\Delta_3$ . The proof is almost exactly as it is for Theorem 7.2 with the addition that injectivity of  $\Psi_\Lambda$  follows from Theorem 4.1 and the injectivity of  $\Pi \circ \Psi_0$  requires a new argument. Our discussion is divided into two cases.

**Case A .**  $S$  has at least two generalized boundary components, say  $\Delta_2, \Delta'_2$ . Since  $S$  is a non-exceptional surface, there is a generalized  $\mathcal{X}$ -piece on  $S$  such that  $\Delta_2, \Delta'_2$  are its two generalized boundary components, and the remaining two generalized boundary components, denoted by  $\Delta_3, \Delta'_3$  are geodesic boundaries. Without loss of generality, we assume that  $\Delta_2, \Delta'_2$  are cone points. Let  $\gamma$  be a waist of this  $\mathcal{X}$ -piece and  $\delta$  be a simple closed curve which separates  $\Delta_2, \Delta'_2$  from  $\Delta_3, \Delta'_3$ . Let  $\beta$  (resp.  $\beta'$ ) be a simple arc connecting  $\Delta_2$  and  $\Delta'_2$  which sits on the pants bounded by  $\Delta_2, \Delta'_2$  and  $\delta$  (connecting  $\Delta_3$  and  $\Delta'_3$  which sits on the pants bounded by  $\Delta_3, \Delta'_3$  and  $\delta$ ). Hence  $\{\delta, \beta, \beta'\}$  consists a coordinate system of curves which gives a Fenchel-Nilsen coordinates of  $X$ . Let  $\delta_n$  be a simple closed curve obtained from  $\delta$  by  $n$  times Dehn twist along  $\gamma$ . Denote by  $a_2$  (resp.  $a'_2$ ) the geodesic arc perpendicular to both  $\Delta_2$  and  $\gamma$  (resp.  $\Delta'_2$  and  $\gamma$ ).

Combining (8) and (10), we have the following formula  
(29)

$$\cosh \frac{l_X([\delta_n])}{2} = \sin\left(\frac{|\lambda_2|}{2}\right) \sin\left(\frac{|\lambda'_2|}{2}\right) \{ \cosh |a_2| \cosh |a'_2| \cosh(t+n)l_X([\gamma]) + \sinh |a_2| \sinh |a'_2| \} - \cos\left(\frac{|\lambda_2|}{2}\right) \cos\left(\frac{|\lambda'_2|}{2}\right),$$

where  $t$  represents the twist of  $X$  with respect to  $\{\delta, \beta, \beta'\}$ .

So we have

$$\lim_{n \rightarrow \infty} \frac{\cosh \frac{l_X([\delta_n])}{2}}{\cosh n l_X([\gamma])} = \sin \frac{|\lambda_2|}{2} \sin \frac{|\lambda'_2|}{2} \cosh |a_2| \cosh |a'_2|,$$

where  $|\lambda_2|, |\lambda'_2|$  represent the cone angles at  $\Delta_2, \Delta'_2$ , respectively.

Hence

$$(30) \quad \lim_{n \rightarrow \infty} e^{\frac{l_X([\delta_n])}{2} - n l_X([\gamma])} = \sin \frac{|\lambda_2|}{2} \sin \frac{|\lambda'_2|}{2} \cosh |a_2| \cosh |a'_2|.$$

The left side and the right side of (30) can be viewed as two functions over  $\mathcal{T}_{g,n}(\Lambda)$ . Denote these two functions by  $F(X), G(X)$  respectively. It follows from (9) that

$$\sinh |a_2| = \frac{\cos \frac{|\lambda_2|}{2} + \cos \frac{|\lambda'_2|}{2} \cosh \frac{l_X(\gamma)}{2}}{\sin \frac{|\lambda_2|}{2} \sinh \frac{l_X(\gamma)}{2}},$$

which means that  $|a_2|$  is a decrease function of  $l_X(\gamma)$ . Similarly,  $|a'_2|$  is a decrease function of  $l_X(\gamma)$ . Therefore  $G(X)$  is a decrease function of  $l_X(\gamma)$ . If there were  $X, X' \in \mathcal{T}_{g,n}(\Lambda)$  and  $k > 1$  such that  $l_{X'}([\alpha]) = k l_X([\alpha])$  holds for any  $[\alpha] \in \mathcal{S}(S)$ , then  $F(X') > F(X)$ . On the other hand,  $G(X)$  is a decrease function of  $l_X(\gamma)$ , then  $G(X') < G(X)$  (note that  $|\lambda'_2|$  and  $|\lambda_2|$  are fixed). But this is impossible since  $F(X) = G(X)$  for any  $X \in \mathcal{T}_{g,n}(\Lambda)$ .

**Case B.** The surface  $S$  has only one generalized boundary component. In this case, if  $S$  has genus at least two, we can choose two non-peripheral simple closed curves  $\gamma_1, \gamma_2$  which cut  $S$  into two pieces, one of which, say  $S_1$ , has one genus and two geodesic boundary components. Let  $X'$  be the restriction of  $X$  on  $S_1$ .  $X'$  is a hyperbolic surface. Following the method used in the proof of [7, Prop.7.12], we obtain the injection of  $\Pi \circ \Phi$ . It remains to consider the case that  $S$  is a torus with only one generalized boundary component  $\Delta_3$ . Let  $\gamma, \beta$  be two simple closed curves on  $S$  such that  $i([\alpha], [\beta]) = 1$  (see Fig. 5(a)), and let  $\beta_n$  be the simple closed curve obtained from  $\beta$  by  $n$  times Dehn twist along  $\gamma$ . From (21), we have

$$\lim_{n \rightarrow \infty} e^{l_X([\beta_n]) - n l_X([\gamma])} = \frac{\cosh^2 \frac{l_X([\gamma])}{2} + \cos \frac{|\lambda_3|}{2}}{\sinh^2 \frac{l_X([\gamma])}{2}},$$

which also implies that there is no  $X' \in \mathcal{T}_{g,n}(\Lambda)$  and  $k > 0$  such that  $l_{X'}([\alpha]) = kl_X([\alpha])$  for any  $[\alpha] \in \mathcal{S}(S)$  (note that  $|\lambda_3|$  is fixed).

(b) Suppose that  $\mathcal{T}_{g,n}(\Lambda) \ni X_n \longrightarrow \xi \in \overline{\partial\mathcal{T}_{g,n}(\Lambda)}^{Th}$ , then for any given  $[\alpha] \in \mathcal{S}(S)$  there exists  $t_n > 0$  such that  $t_n l_{X_n}([\alpha]) \rightarrow \xi([\alpha])$  as  $n \rightarrow \infty$ .

We claim that  $t_n \rightarrow 0$  as  $n \rightarrow \infty$ . Note that  $X_n$  leaves every compact subset of  $\mathcal{T}_{g,n}(\Lambda)$ , it follows from the properness of  $\Psi_\Lambda$  that there is at least one  $[\alpha] \in \mathcal{S}(S)$  such that  $l_{X_n}([\alpha]) \rightarrow \infty$  as  $n \rightarrow \infty$ , which proves the claim since  $t_n l_{X_n}([\alpha]) \rightarrow \xi([\alpha])$  for a finite number  $\xi([\alpha])$ .

Combining this with the estimates in Theorem B, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} t_n l_{F_{\Gamma,\Lambda}(X_n)}([\alpha]) \\ &= \lim_{n \rightarrow \infty} t_n l_{X_n}([\alpha]) + \lim_{n \rightarrow \infty} t_n [l_{F_{\Gamma,\Lambda}(X_n)}([\alpha]) - l_{X_n}([\alpha])] \\ &= \lim_{n \rightarrow \infty} t_n l_{X_n}([\alpha]) \\ &= \xi([\alpha]), \end{aligned}$$

which means that  $F_{\Gamma,\Lambda}(X_n) \longrightarrow \xi$  as  $n \rightarrow \infty$ . From Theorem 7.2, we get  $\overline{\partial\mathcal{T}_{g,n}(0)}^{Th} = \mathcal{PML}(S)$ . Therefore  $F_{\Gamma,\Lambda}(X_n) \longrightarrow \xi \in \mathcal{PML}(S)$ . By interchanging the roles of  $X_n$  and  $F_{\Gamma,\Lambda}(X_n)$ , we get the inverse direction statement, that is, if  $F_{\Gamma,\Lambda}X_n \longrightarrow \xi \in \mathcal{PML}(S)$ , then  $X_n \longrightarrow \xi \in \overline{\partial\mathcal{T}_{g,n}(\Lambda)}^{Th}$ . □

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